Sustainable Retirement Spending

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Sustainable retirement spending: the Czech case

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2 Analytical Model
3 Monte Carlo Simulation Study
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Motivation

Population ageing phenomenon
- Decreasing fertility
- Increasing longevity

Social insurance systems predicted to face sustainability problems
- Financed via pay-as-you-go scheme

Future retirees relying on their own lifetime savings to finance retirement consumption
- Need for informed decisions

Old age dependency ratio [%]

Source: Eurostat, 2011
2 distinct ways to transform savings into periodic income stream...

**Buying a life annuity**

- The easiest way to secure a life-contingent cash flow
- Income stream can not be outlived
- Loss of liquidity/flexibility
- Not possible to leave a bequest
- Theoretically should maximize consumer’s utility – Yaari (1965)
- Actual worldwide demand for annuities is rather low - “annuity puzzle”

**“Self-annuitization”**

- i.e. discretionary management of pension funds with periodic withdrawals for purposes of consumption
- Flexibility
- Possibility of leaving a bequest
- Risk that the pension capital runs out prior to death

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6
2 distinct ways to transform savings into periodic income stream...

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<th>“Self-annuitization”</th>
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</tr>
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<td>• Flexibility ✓</td>
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Key variables of retirement planning

Sustainability of retirement plan

- Spending rates
- Mortality (stochastic)
- Investment returns (stochastic)

We aim to:

- Link these three key factors in a simple analytic model where the probability that a given retirement plan is sustainable can be determined.
- Complement to this probability is the PoR (probability of ruin).
Connection to classical ruin theory in insurance

<table>
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<th>Classical ruin theory</th>
<th>Sustainable spending model</th>
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<td>From perspective of an insurer</td>
<td>From a consumer perspective</td>
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<tr>
<td>Deterministic income: premium</td>
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</tr>
<tr>
<td>Stochastic outcome: claims</td>
<td>Deterministic outcome: consumption</td>
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</tbody>
</table>

- Premium
- Claims
- Investment returns
- Consumption
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Simplified deterministic situation

**Assume:**
- \( r \) … fixed inflation-free interest rate
- \( T \) … fixed remaining length of life
- \( c \) … desired lifelong consumption stream (\( c \) real CZK per annum)
- \( w \) … initial capital (retirement savings)

**Then:**

Present value of desired future consumption (PV):

\[
c \bar{a}_T = c \int_0^T e^{-rt} \, dt
\]

Probability of ruin:
- \( PoR(w) = 1 \), if \( w < PV \)
- \( PoR(w) = 0 \), if \( w \geq PV \)
Stochastic model for mortality

Gompertz-Makeham law of mortality:

\[ \lambda(x) = \lambda + \frac{1}{b} \exp \left\{ \frac{x - m}{b} \right\}, \quad x \geq 0 \]

- \( \lambda(x) \) … instantaneous force of mortality at age \( x \)
- \( m > 0 \) … location parameter
- \( b > 0 \) … scale parameter
- \( \lambda \geq 0 \) … component attributable to accidental deaths

Probability of survival to age \( x+t \), conditional on a life at age \( x \):

\[ t p_x = P[T_x > t] = \exp \left\{ -\lambda t + e^{(x-m)/b} \left( 1 - e^{t/b} \right) \right\} \]

- \( T_x \) … remaining lifetime at age \( x \)

Figure:
Density of \( T_x \)

\[ x=0, \lambda=0, m=83, b=11 \]

\[ x=65, \lambda=0, m=83, b=11 \]
Stochastic model for investment returns

We assume the standard geometric Brownian motion (GBM) model of market prices, i.e. the real value of investment portfolio obeys the SDE:

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \quad S_0 = 1 \]

- \( S_t \) ... real (after-inflation) portfolio value at time \( t \)
- \( B_t \) ... standard Brownian motion (Wiener process)
- \( \mu \) ... rate of return (drift coefficient)
- \( \sigma \) ... volatility (diffusion coefficient)

Or equivalently:

\[ S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t} \]

Growth condition:

\[ \mu - \sigma^2/2 > 0 \]
Brownian motion - illustration

Standard Brownian motion

- $B_t \sim N(0, t)$

Geometric Brownian motion

- $S_t = S_0 \exp\{\nu t + \sigma B_t\}$
- $S_t|S_0 \sim LN(\log S_0 + \nu t, \sigma^2 t)$

Non-standard Brownian motion

- $B_t^{(\nu, \sigma)} = \nu t + \sigma B_t$
- $B_t^{(\nu, \sigma)} \sim N(\nu t, \sigma^2 t)$

$\nu = 0.05, \sigma = 0.02$

$\nu = 0.02, \sigma = 0.3, S_0 = 1$

$\nu = 0.1, \sigma = 0.1$
Net wealth process SDE

If the retiree invests his initial capital \( w \) in a portfolio satisfying the SDE
\[
dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \quad S_0 = 1
\]
and than consumes a fixed real amount \( k \) per year, than his net wealth process \( W_t \) obeys the following SDE:
\[
dW_t = (\mu W_t - k) \, dt + \sigma W_t \, dB_t, \quad W_0 = w.
\]

This equation can be solved using a stochastic analogy to the variation of coefficients method (see e.g. Karatzas & Shreve, 1991) to yield an explicit formula for the net wealth process:
\[
W_t = S_t \left[ w - k \int_0^t \frac{1}{S_u} \, du \right], \quad t \geq 0.
\]

Observe that for all \( T \in (0, \infty] \) it holds that
\[
P \left[ \inf_{0 \leq t \leq T} W_t \leq 0 \middle| W_0 = w \right] = P \left[ W_T \leq 0 \middle| W_0 = w \right].
\]
Reciprocal gamma distribution

Random variable $Y$ obeys the *reciprocal gamma (RG) distribution* with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, denoted by $Y \sim RG(\alpha, \beta)$, if $Y = \frac{1}{X}$, where $X \sim Gamma(\alpha, \beta)$.

Consequently, we have for any $x > 0$:

$$G(x; \alpha, \beta) = 1 - G_R\left(\frac{1}{x}; \alpha, \beta\right)$$

- $G(x; \alpha, \beta)$ … CDF of gamma distribution
- $G_R(x; \alpha, \beta)$ … CDF of reciprocal gamma gamma distribution.

The first two moments of the RG distribution are:

$$E[Y] = \frac{1}{\beta(\alpha - 1)},$$

$$E[Y^2] = \frac{1}{\beta^2(\alpha - 1)(\alpha - 2)}$$
Eventual probability of ruin (EPoR) (1/2)

\[ EPoR = \text{probability that the net wealth process will ever hit zero:} \]

\[ EPoR(w) = P\left[ \inf_{0\leq t \leq \infty} W_t \leq 0 \mid W_0 = w \right] \]

\[ = P \left[ W_\infty \leq 0 \mid W_0 = w \right] \]

\[ = P \left[ \frac{w}{k} \leq \int_0^\infty \frac{1}{S_u} du \right] \]

\[ =: Z \]

- \( Z \) … present value of a stochastic perpetuity (PVSP)

Theorem:
The PVSP random variable obeys RG distribution: \( Z \sim RG\left(\frac{2\mu}{\sigma^2} - 1, \frac{\sigma^2}{2}\right) \)

Eventual probability of ruin (EPoR) (2/2)

Since the PVSP random variable $Z$ follows the reciprocal gamma distribution, we can calculate the eventual probability of ruin easily by evaluating the CDF of gamma distribution at $\frac{k}{w}$:

$$\text{EPoR} (w) = P \left[ \frac{w}{k} \leq Z \right] = 1 - G_R \left( \frac{w}{k}; \frac{2\mu}{\sigma^2} - 1, \frac{\sigma^2}{2} \right) = G \left( \frac{k}{w}; \frac{2\mu}{\sigma^2} - 1, \frac{\sigma^2}{2} \right)$$

Note: Gamma distribution is available in MS Excel.
**Lifetime probability of ruin (LPoR)**

$LPoR = \text{probability that the net wealth process will hit zero while the retiree is still living.}$

We have:

$$LPoR(w) = P \left[ \inf_{0 \leq t \leq T_x} W_t \leq 0 \mid W_0 = w \right]$$

$$= P \left[ W_{T_x} \leq 0 \mid W_0 = w \right]$$

$$= P \left[ \frac{w}{k} \leq \int_0^{T_x} \frac{1}{S_u} \, du \right]$$

- $Z_{T_x}$ … stochastic present value (SPV) of a life annuity

There is no closed-form density function for SPV. However, its distribution can be estimated…
Reciprocal gamma approximation (1/3)

SPV random variable:

\[ Z_{T_x} := \int_{0}^{T_x} e^{-(\mu - \sigma^2/2)s - \sigma B_s} \, ds \]

True distribution of \( Z_{T_x} \): unknown

Approximation technique: moment matching

Candidate distribution: reciprocal gamma distribution (2 degrees of freedom)

Assumptions used: \( T_x \) independent of the Brownian motion \( \{B_t, t \geq 0\} \) driving \( S_u \)

Recall the first two moments of \( Y \sim RG(\alpha, \beta) \):

\[ M_1 = \mathbb{E}[Y] = \frac{1}{\beta(\alpha - 1)} \]
\[ M_2 = \mathbb{E}[Y^2] = \frac{1}{\beta^2(\alpha - 1)(\alpha - 2)} \]

Solving for \( \alpha \) and \( \beta \) yields:

\[ \alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \beta = \frac{M_2 - M_1^2}{M_2M_1} \]
Reciprocal gamma approximation (2/3)

Calculate the first two moments of $Z_{TX}$:

$$\tilde{M}_1 = \mathbb{E} \left[ Z_{TX} \right] = \cdots = A(\mu - \sigma^2 | \cdot)$$

$$\tilde{M}_2 = \mathbb{E} \left[ Z_{TX}^2 \right] = \cdots = \frac{2}{\mu - 2\sigma^2} \left[ A(\mu - \sigma^2 | \cdot) - A(2\mu - 3\sigma^2 | \cdot) \right]$$

where we defined

$$A(\xi | \cdot) := \int_0^\infty \exp \left\{ -\xi s \right\} s \varphi_x \, ds$$

as the price of a continuous life annuity under a continuously compounded interest rate $\xi$.

In case $T_x$ follows the Gompertz-Makeham law of mortality, we have

$$A(\xi | \lambda, m, b, x) = b \exp \left\{ e^{(x-m)/b} + (\xi + \lambda)(x - m) \right\} \Gamma(-(\xi + \lambda)b, e^{(x-m)/b})$$

where

$$\Gamma(a, c) = \int_c^\infty e^{-t} t^{a-1} \, dt$$

stands for the (upper) incomplete Gamma function.
Reciprocal gamma approximation (3/3)

Now the fitted parameter values of RG distribution can be calculated as

\[ \hat{\alpha} = \frac{2\tilde{M}_2 - \tilde{M}_1^2}{\tilde{M}_2 - \tilde{M}_1^2}, \quad \hat{\beta} = \frac{\tilde{M}_2 - \tilde{M}_1^2}{\tilde{M}_2 \tilde{M}_1} \]

and we can conclude that the approximate distribution of the SPV random variable \( Z_{T_x} \) is

\[ Z_{T_x} \sim RG(\hat{\alpha}, \hat{\beta}). \]

Finally, we can express the approximated lifetime probability of ruin:

\[ \text{LPoR}(w) = P \left[ \frac{w}{k} \leq Z_{T_x} \right] \]

\[ \approx 1 - G_R \left( \frac{w}{k}; \hat{\alpha}, \hat{\beta} \right) = G \left( \frac{k}{w}; \hat{\alpha}, \hat{\beta} \right) \]
Figure 4.1: The RG approximation of the LPoR as a function of the investment volatility $\sigma$, for various values of the expected real rate of return $\mu$. A 65-year-old individual with an initial wealth $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. 
Sustainable spending rate

The RG formula for life time probability of ruin

$$\text{LPoR}(w) \approx G\left(\frac{k}{w}; \hat{\alpha}, \hat{\beta}\right)$$

can be inverted to yield the so-called maximal sustainable spending rate (SSR):

$$k \approx w \cdot G^{-1}\left(\text{LPoR}(w); \hat{\alpha}, \hat{\beta}\right).$$

Interpretation:
For a fixed initial capital $w$, SSR is the maximal annual spending rate, so that the corresponding consumption plan results in a chosen tolerated probability of ruin $\text{LPoR}(w)$.

Note:
$G^{-1}(\cdot; \alpha, \beta)$ … quantile function of gamma distribution (available in MS Excel)
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Monte Carlo simulations

1. • Generate time of death $T_x$
   • Sampling from Gompertz distribution

2. • Generate path of the net wealth process $W_t$
   • Repeated sampling from normal distribution

3. • Decide if ruin of the individual occurred
   • Did $W_t$ cross zero for some $t_0<T_x$?

- For a given set of parameters independently repeat steps 1. to 3. for $n$ times
- Monte Carlo LPoR approximation:

$$MC\ LPoR = \frac{\# \ observed\ events\ of\ ruin}{n}$$

- In the following analysis: $n = 10000$
Convergence of the MC method

Figure 5.1: The speed of convergence of the Monte Carlo LPoR approximation. The figure displays the MC approximation of the LPoR as a function of the investment volatility $\sigma$, for various number of simulations $n$. A 65-year-old individual with an initial wealth $w = 10$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The assumed expected real rate of investment return is $\mu = 0.1$. 
Figure 5.2: The MC approximation of the LPoR as a function of the investment volatility $\sigma$, for various values of the expected real rate of return $\mu$. A 65-year-old individual with an initial wealth $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. 
Figure 5.3: The discrepancy between the RG and the MC approximations of the LPoR as a function of the investment volatility $\sigma$, for various values of the expected real rate of return $\mu$. A 65-year-old individual with an initial wealth $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. 
Figure 5.4: The discrepancy between the RG and the MC approximations of the LPoR as a function of the investment volatility $\sigma$, for various values of the initial wealth $w$. A 65-year-old individual, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$. 
Difference between the RG and MC approximation (3/3)

Figure 5.5: The discrepancy between the RG and the MC approximations of the LPoR as a function of the investment volatility $\sigma$, for various values of the age $x$ of the retiree. An individual with the initial wealth of $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$. 
In general, the RG approximation is reasonably accurate as long as $\sigma$ does not increase beyond a certain level, at which the approximation starts to overrate the $LPoR$.

Recall the second moment of SPV:

$$E \left[ Z_{T_x}^2 \right] = \frac{2}{\mu - 2\sigma^2} \int_0^\infty \left( e^{-(\mu - \sigma^2)s} - e^{-(2\mu - 3\sigma^2)s} \right) s p_x \, ds.$$  

In case of high $\sigma$ relative to $\mu$, the second moment of SPV becomes large because of the second exponential term in the integrand and hence the moment matching approximation deteriorates. Due to the $sp_x$ factor, the exact threshold depends also on $x$ and on mortality law parameters.

**Rule of thumb:**

The RG approximation of $LPoR$ deteriorates for $\sigma > \sqrt{2\mu/3}$. 
Benefits of RG approximation over Monte Carlo

- **Monte Carlo Simulations**
  - Accuracy
  - Flexibility

- **Reciprocal Gamma Approximation**
  - Easy to implement
  - Lower time complexity
  - Analytically tractable
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Estimation of Gompertz parameters

Define loss function:

\[
L (m, b) := \sum_{x=x_0}^{\omega} \sqrt{D_x} \left| 1 - \frac{q_x(m, b)}{\hat{q}_x} \right|
\]

- \( \hat{q}_x, \ x = x_0, ..., \omega \) ... death rate at age \( x \) (life table, \( \omega \) ... ultimate age)
- \( D_x, \ x = x_0, ..., \omega \) ... total amount of death claims associated with rate \( \hat{q}_x \)
- \( \hat{q}_x(m, b) \) ... analytic Gompertz death rate at age \( x \)

and set:

\[
(\hat{m}, \hat{b}) := \arg \min_{m \geq 0, b \geq 0} L (m, b)
\]

Results:

Life table Czech Republic (2011), \( x_0 = 60, \ \omega = 105 \)
- female: \( \hat{m}_F = 87.9, \ \hat{b}_F = 7.6 \)
- male: \( \hat{m}_M = 82.5, \ \hat{b}_M = 10.5 \)
Figure 6.2: An illustration of the Gompertz law fit to the female life table (Czech Republic, 2011). The estimated parameter values are $\hat{m}_F = 87.87$ and $\hat{b}_F = 7.64$. 
Fit to male life table (Czech Republic, 2011)

Figure 6.1: An illustration of the Gompertz law fit to the male life table (Czech Republic, 2011). The estimated parameter values are $\hat{m}_M = 82.51$ and $\hat{b}_M = 10.54$. 
Estimated density of remaining lifetime

Gompertz law of mortality
- Life table Czech Republic (2011)
- Female: $m = 87.9$, $b = 7.7$
- Male: $m = 82.5$, $b = 10.5$

**Figure:** density of $T_x$
Financial portfolio constructed by combination of 2 assets

Reinvestment Savings Government Bond of the Czech Republic

- Issued by the Ministry of Finance
- 5-year maturity; tranche from May, 2013
- Fixed avg. real rate of return: $\mu_B = 1.074\%$

PX index (official index of the Prague Stock Exchange)

- Historical (after-inflation) prices observed between 2/1/2003 and 28/12/2012
- Expected return and volatility estimated by moment matching technique (see e.g. Cipra, 2008)
  - $\hat{\mu}_{PX} = 8.79\%$, $\hat{\sigma}_{PX} = 24.5\%$

<table>
<thead>
<tr>
<th>Allocation to bonds</th>
<th>$\mu$ (drift)</th>
<th>$\sigma$ (volatility)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.08787</td>
<td>0.24475</td>
</tr>
<tr>
<td>20%</td>
<td>0.07244</td>
<td>0.19580</td>
</tr>
<tr>
<td>40%</td>
<td>0.05702</td>
<td>0.14685</td>
</tr>
<tr>
<td>60%</td>
<td>0.04159</td>
<td>0.09790</td>
</tr>
<tr>
<td>80%</td>
<td>0.02616</td>
<td>0.04895</td>
</tr>
<tr>
<td>100%</td>
<td>0.01074</td>
<td>0.00000</td>
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*Note: The residual is allocated to the PX index.*
Fig. 3 Development of the daily Prague Exchange (PX) index price over the last ten years (2/1/2003–28/12/2012). The real price was calculated by subtracting the (monthly) inflation from the nominal price. 
Figure: Sustainable spending rate in the Czech Republic

Sustainable spending rate in Czech Republic
(per 100 CZK of initial capital, portfolio = 100% PX index)
### Table: Sustainable spending rate in the Czech Republic

<table>
<thead>
<tr>
<th>Age at retirement</th>
<th>Tolerated probability of ruin</th>
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<tr>
<td></td>
<td>1%</td>
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<tr>
<td></td>
<td>M</td>
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<td>79</td>
<td>4.51</td>
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The table displays the maximal real annual spending ($k$) [in CZK] per 100 CZK of initial wealth ($w$) for a given tolerated lifetime probability of ruin. An investment portfolio with initial 60% allocation to retail savings bonds and 40% to PX index was assumed.

$M$ male, $F$ female. The values are in CZK.
Impact of investment strategy on the LPoR for various spending rates (per 100 CZK of initial capital)

Initial allocation to retail savings bonds (remainder is allocated to PX index)
Optimal investment allocation for retiring Czech females

<table>
<thead>
<tr>
<th>Age at retirement</th>
<th>PoR = 1 %</th>
<th>PoR = 5 %</th>
<th>PoR = 10 %</th>
<th>PoR = 20 %</th>
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<tr>
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<td>SSR</td>
<td>RSB (%)</td>
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For a given age and tolerated PoR the table displays an optimal initial allocation to retail savings bonds and the corresponding sustainable spending rate (SSR). The rest of the portfolio is allocated to the PX index. The SSR values are in Czech crowns (CZK) per 100 CZK of initial capital.

*RSB* Retail Savings Bonds, *SSR* Sustainable Spending Rate
Optimal investment allocation for retiring Czech males

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<th>PoR = 1 %</th>
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For a given age and tolerated PoR the table displays an optimal initial allocation to retail savings bonds and the corresponding sustainable spending rate. The rest of the portfolio is allocated to the PX index. The SSR values are in CZK per 100 CZK of initial capital.

*RSB Retail Savings Bonds, SSR Sustainable Spending Rate*
1 Introduction
2 Analytical Model
3 Monte Carlo Simulation Study
4 Case Study with Czech Data
5 Conclusion
Summary

Monte Carlo Simulations

Reciprocal Gamma Approximation

Sustainability of retirement plan

Spending rates

Mortality

Investment returns

Accuracy

Flexibility

Easy to implement

Lower time complexity

Analytically tractable

Graphs and charts illustrating the impact of initial allocation to retail savings bonds on longevity-based purchase rate (LpR), with varying parameters. Graphs also show the relationship between volatility and the difference between the longevity-based purchase rate and the Monte Carlo LpR.
References