KPMG

cutting through complexity

Chain-ladder extensions

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Contents

Introduction

Reserving methods “in practice” based on triangles

- Chain-ladder
  - Triangle of paid claims
  - Triangle of incurred claims
  - Triangle of reported claims
  - *Triangle of incurred counts*
- Münich chain-ladder
  - Triangle of paid claims + Triangle of incurred claims

Goals

- Best estimate
- Mean square error of prediction
- VaR 99.5%
- Other characteristics
- Full distribution
  - Fitting of chosen distribution to first two moments
  - Bootstrap

Aggregation of data

Triangles aggregate data

+ Convenient presentation

– Loss of information which in some cases may lead to a poor performance

Individual claims modeling

+ No loss of information

– Usually complex models with lots of parameters

– Require large datasets (which might not be available)

– Might be computationally expensive

Trade-off

Having simple model vs. Using all information

1st version of the proposed model

„Key ideas“

Prediction of RBNS and IBNR claims using claim amounts and claim counts

R. Verrall, J. P. Nielsen, A. H. Jessen

April 2010

1st version of the model

Basic ideas in comparison with chain-ladder

Chain-ladder

- Based on one triangle (paid / incurred / reported)
- All sources of delay (reporting, payment) incorporated in one development pattern

Proposed alternative

- Basic idea is to separate the sources of delay ➔ using more than one triangle
  - Triangle of incurred counts → reporting delay
  - Triangle of claims paid → payment delay
- *Using triangle of incurred claims as a further supplementary source of information considered in BDCL model*

1st version of the model
Basic ideas in comparison with chain-ladder (cont'd)

Chain-ladder

- There was an algorithm without an underlying stochastic model
- Underlying stochastic models added later
  - Poisson model (CL is maximum-likelihood estimator)
  - **Mack distribution-free model**
  - …

Proposed alternative

- First, there is an underlying exact compound Poisson model based on more detailed data
- Proposed model to be used in practice – *double chain-ladder* – is its approximation

1^{st} version of the model

Formal structure

$\Delta_m = (X_{ij} : 1 \leq i+j \leq m)$ **traingle of claims paid**

$\aleph_m = (N_{ij} : 1 \leq i+j \leq m)$ **traingle of incurred claims counts**

- Claim is not usually paid immediately after notification. This motivates the introduction of the third triangle.

N_{ijk}^{paid} **– part of the N_{ij} claims fully paid with k periods delay after being reported,** $k = 0, \ldots, d$; d **is max. delay**

N_{ij}^{paid} **– number of claims incurred in period i and (fully) paid with j periods delay (new triangle)**

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

Note, that this last triangle plays an important role in the derivation of the model but, nevertheless, it is not assumed to be known.

Assumptions

- N_{ij} independent, with over-dispersed Poisson distribution (ML estimate leads to classical CL algorithm)
- Given N_{ij}, the distribution of the numbers of paid claims follows a multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

- Claim settled with one payment. Thus, if we denote $Y_{ij}(k)$ the payment for the k-th claim incurred in period i settled with j periods delay, we have

$$X_{ij} = Y_{ij}(1) + Y_{ij}(2) + \ldots + Y_{ij}(N_{ij}^{paid})$$

- $Y_{ij}(k)$ i.i.d., independent of number of claims, independent of reporting and payment delay (authors were aware that this is probably not valid in practice)

1st version of the model
Derivation

"Maximum-likelihood estimate"

Likelihood function

$$\mathcal{L}_{\aleph_m,\triangle_m} = \mathcal{L}_{\aleph_m} \times \mathcal{L}_{\triangle_m|\aleph_m}$$

$$= \left(\prod_{i=1}^{m} \prod_{j=0}^{m-i} P(N_{ij} = n_{ij}) \right)$$

$$\times \left(\prod_{i=1}^{m} f_{X_{i0},\dots,X_{i,m-i}|N_{i0},\dots,N_{i,m-i}}(x_{i0},\dots,x_{i,m-i}|n_{i0},\dots,n_{i,m-i}) \right)$$

Functions of different parameters – can be maximized separately

- The first one is maximized with CL algorithm on the triangle $\aleph_m$ of incurred claim counts
- Not obvious how to maximize the second (at least, we did not specify distributional assumptions about payments)
  - Proposed approximation of the model
  - Construct quasi-log likelihood which requires just the first two moments

Mean

$$
\begin{aligned}
E[X_{ij}|\aleph_m] &= E[E[X_{ij}|N_{ij}^{paid}]|\aleph_m] \\
&= E\left[E\left[\sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}|N_{ij}^{paid}\right]|\aleph_m\right] \\
&= E[N_{ij}^{paid}E[Y_{ij}^{(k)}]|\aleph_m] \\
&= E[N_{ij}^{paid}|\aleph_m]E[Y_{ij}^{(k)}]
\end{aligned}
$$

Variance

$$
\begin{aligned}
V[X_{ij}|\aleph_m] &= E[V[X_{ij}|N_{ij}^{paid}]|\aleph_m] + V[E[X_{ij}|N_{ij}^{paid}]|\aleph_m] \\
&= E[V[\sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}|N_{ij}^{paid}]|\aleph_m] + V[N_{ij}^{paid}E[Y_{ij}^{(k)}]|\aleph_m] \\
&= E[N_{ij}^{paid}V[Y_{ij}^{(k)}]|\aleph_m] + V[N_{ij}^{paid}E[Y_{ij}^{(k)}]|\aleph_m]
\end{aligned}
$$

1^{st} version of the model

Derivation (cont'd)

Since we assume that $Y_{ij}(k)$ are i.i.d., we have

$$E[Y_{ij}(k)] = \mu, \qquad V[Y_{ij}(k)] = \sigma^2$$

Thus

$$E[X_{ij}|\aleph_m] = E[N_{ij}^{paid}|\aleph_m]\mu$$

$$V[X_{ij}|\aleph_m] = E[N_{ij}^{paid}|\aleph_m]\sigma^2 + V[N_{ij}^{paid}|\aleph_m]\mu^2$$

Using the assumption of conditional multinomial distribution of N_{ij}^{paid}

$$\begin{aligned} E[N_{ij}^{paid}|\aleph_m] &= E\left[\sum_{k=0}^{\min\{j,d\}} N_{i,j-k,k}^{paid}|\aleph_m\right] \\ &= \sum_{k=0}^{\min\{j,d\}} E[N_{i,j-k,k}^{paid}|\aleph_m] \\ &= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k}p_k \end{aligned}$$

Assuming that the numbers of claims paid from different origin years are uncorrelated

$$
\begin{aligned}
V[N_{ij}^{paid}|\aleph_m] &= V\left[\sum_{k=0}^{\min\{j,d\}} N_{i,j-k,k}^{paid}|\aleph_m\right] \\
&= \sum_{k=0}^{\min\{j,d\}} V[N_{i,j-k,k}^{paid}|\aleph_m] \\
&= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (1-p_k)
\end{aligned}
$$

Hence

$$E[X_{ij}|\aleph_m] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \mu$$

$$\begin{aligned} V[X_{ij}|\aleph_m] &= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \sigma^2 + \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (1-p_k) \mu^2 \\ &= \sum_{k=0}^{\min(j,d)} N_{i,j-k} \{\sigma^2 p_k + \mu^2 p_k (1 - p_k)\} \\ &\approx \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (\sigma^2 + \mu^2) \end{aligned}$$

Last approximation is done so that the variance is proportional to the mean

➔ An over-dispersed Poisson model may be used.

This leads to the proposed algorithm

- Apply chain-ladder to the triangle of the incurred claims counts (needed for the IBNR claims only)
- Fit the over-dispersed Poisson model to the paid claims triangle with mean

$$E[X_{ij}|\aleph_m] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \mu = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} \psi_k$$

  from which ML estimates of ψ_k can be derived
- Compute estimates of μ and p_k from formulas

$$\sum_{k=0}^{d} \psi_k = \sum_{k=0}^{d} \mu p_k = \mu \sum_{k=0}^{d} p_k = \mu \qquad \psi_k = \mu p_k$$

- **Estimate claims reserves – separately for reported and not yet reported claims**

Reported claims $\mu \sum_{k=i-m+j}^{\min\{j,d\}} p_k N_{i,j-k}$ IBNR claims $\mu \sum_{k=0}^{\min\{i-m+j-1,d\}} p_k \hat{N}_{i,j-k}$

- Variance can also be estimated using the estimate of the over-dispersion parameter

Triangle of counts

i\j	0	1	2	3	4	5	6	7	8	9
1	6238	831	49	7	1	1	2	1	2	3
2	7773	1381	23	4	1	3	1	1	3	
3	10306	1093	17	5	2	0	2	2		
4	9639	995	17	6	1	5	4			
5	9511	1386	39	4	6	5				
6	10023	1342	31	16	9					
7	9834	1424	59	24						
8	10899	1503	84							
9	11954	1704								
10	10989									

1st version of the model

Case study provided in the paper (MTPL)

Triangle of paid claims (adjusted to calendar inflation)

i\j	0	1	2	3	4	5	6	7	8	9
1	451288	339519	333371	144988	93243	45511	25217	20406	31482	1729
2	448627	512882	168467	130674	56044	33397	56071	26522	14346	
3	693574	497737	202272	120753	125046	37154	27608	17864		
4	652043	546406	244474	200896	106802	106753	63688			
5	566082	503970	217838	145181	165519	91313				
6	606606	562543	227374	153551	132743					
7	536976	472525	154205	150564						
8	554833	590880	300964							
9	537238	701111								
10	684944									

1st version of the model

Remarks

Case study in the paper

- Adjustment for zero-claim is applied: $P(Y_{ij}^{(k)} = 0) = Q$, where Q set by expert judgment.
- Results – only best estimate available (MSEP, full distribution estimates etc. not considered in the paper)
  - Difference in the total best estimate is not large.
  - However, in the following paper it was suggested that using more data should imply less volatility (thus lower solvency requirement corresponding to VaR 99.5%)

i	IBNR	RBNS	TOTAL	CHAIN LADDER
2	628	605	1,233	1,685
3	1,350	4,514	5,863	29,379
4	1,510	43,623	45,133	60,638
5	1,967	94,526	96,493	101,158
6	2,579	171,633	174,212	173,802
7	3,168	299,136	302,304	249,349
8	5,349	509,334	514,684	475,992
9	14,280	852,144	866,423	763,919
10	254,499	1,135,678	1,390,177	1,459,860
Total	285,329	3,111,192	3,396,521	3,315,779

2^{nd} version of the proposed model

Bootstrap

Cash flow simulation for a model of outstanding liabilities based on claim amounts and claim numbers

M. D. Martínez-Miranda, B. Nielsen,
J. P. Nielsen, R. Verrall

September 2010

This leads to the proposed algorithm

- Apply chain-ladder to the triangle of the incurred claims counts (needed for the IBNR claims only)
- Fit the over-dispersed Poisson model to the paid claims triangle with mean

$$E[X_{ij}|\aleph_m] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \mu = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} \psi_k$$

  from which ML estimates of ψ_k can be derived
- Compute estimates of μ and p_k from formulas

$$\sum_{k=0}^{d} \psi_k = \sum_{k=0}^{d} \mu p_k = \mu \sum_{k=0}^{d} p_k = \mu$$

$$\psi_k = \mu p_k$$

- **Estimate claims reserves – separately for reported and not yet reported claims**

Reported claims $$\mu \sum_{k=i-m+j}^{\min\{j,d\}} p_k N_{i,j-k}$$

IBNR claims $$\mu \sum_{k=0}^{\min\{i-m+j-1,d\}} p_k \hat{N}_{i,j-k}$$

- Variance can also be estimated using the estimate of the over-dispersion parameter

2nd version of the model

Proposed alteration from the 1st model

Recall a key step in the first version of the model

- Parameters ψ_k estimated by fitting ODP model to the claims paid triangle with mean

$$\mathsf{m}_{ij}(N) = \mathsf{E}(X_{ij} \mid N) = \sum_{k=0}^{\min(j,d)} N_{i,j-k}\,\mu p_k$$

- Fitting done by maximizing (pseudo log-)likelihood function (index *I* means known triangle)

$$\ell^{pseudo}(\psi; X, N) = \sum_{i,j \in \mathcal{I}} \{X_{ij} \log \mathsf{m}_{ij}(N) - \mathsf{m}_{ij}(N)\}$$

- No closed form solution – must be done numerically. Technical difficulties may arise, for example:
  - *Numerical procedure may give negative* ψ_k
  - ***May be computationally intensive – potential drawback for bootstrapping***

➔ Suggestion: approximation allowing for estimate by an analytical formula

2^{nd} version of the model

Derivation of parameter estimates

Approximation: replace known N_{ij} by estimated counts from the chain-ladder algorithm.

- *Naturally, this "requires" that these estimates are not far from observed counts*
- *Requires d = m – 1 (i.e. maximum delay corresponds to the dimension of the triangle)*

Recall that for chain-ladder development factors, we have

$$F_\ell = \frac{\sum_{i=1}^{m-\ell} \sum_{j=0}^{\ell} N_{ij}}{\sum_{i=1}^{m-\ell} \sum_{j=0}^{\ell-1} N_{ij}}, \quad 1 \le \ell \le m-1$$

We define the ratios

$$\widehat{B}_j = \widehat{N}_{ij} / \widehat{N}_{i0} = \begin{cases} (F_j - 1) \prod_{k=1}^{j-1} F_k & j \ge 2, \\ F_1 - 1 & j = 1, \\ 1 & j = 0, \end{cases}$$

Here, hats are used to denote values estimated by the chain-ladder algorithm.

See, that the ratios do not depend on *i*, the accident period index.

We replace $m_{ij}(N)$ in the (psuedo log-)likelihood function by

$$\mathrm{m}_{ij}(\widehat{N}) = \sum_{k=0}^{j} \widehat{N}_{i,j-k}\psi_k = \widehat{N}_{i0}\zeta_j$$

where

$$\zeta_j = \sum_{k=0}^{j} \widehat{B}_{j-k}\psi_k$$

and we get

$$\ell^{pseudo}(\psi; X, \widehat{N}) = \sum_{i,j\in\mathcal{I}} X_{ij}\log(\widehat{N}_{i0}) + \sum_{j=1}^{m-1}\{\log(\zeta_j)\sum_{i=1}^{m-j} X_{ij} - \zeta_j \sum_{i=1}^{m-j}\widehat{N}_{i0}\}$$

Now, this function, with variables ζ_j, can be maximized analytically (taking partial derivatives equal to 0, etc.)

The solution is

$$\widehat{\zeta}_j = \frac{\sum_{i=1}^{m-j} X_{ij}}{\sum_{i=1}^{m-j}\widehat{N}_{i0}}$$

2nd version of the model

Derivation of parameter estimates (cont'd)

Since we have

$$\zeta_j = \sum_{k=0}^{j} \widehat{B}_{j-k}\psi_k$$

and we derived estimates for the left side, the estimates of ψ_k can be derived by solving the linear system

$$\begin{pmatrix} \widehat{\zeta}_0 \\ \vdots \\ \vdots \\ \widehat{\zeta}_{m-1} \end{pmatrix} = \begin{pmatrix} \widehat{B}_0 & 0 & \cdots & 0 \\ \widehat{B}_1 & \widehat{B}_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \widehat{B}_{m-1} & \cdots & \widehat{B}_1 & \widehat{B}_0 \end{pmatrix} \begin{pmatrix} \widehat{\psi}_0 \\ \vdots \\ \vdots \\ \widehat{\psi}_{m-1} \end{pmatrix}$$

Again, this can result in negative estimates of ψ_k. Authors suggested:

- If the sum of absolute values of negative ψ_k estimates is under 1% then replace them by zero (and adjust other factors proportionally)
- If the sum is larger then consider adjustment for zero claims

Best estimate of reserves – same formulas as in the first version of the model

(parameters estimates are done differently)

RBNS part

$$\tilde{\mathsf{m}}_{ij}(N) = \sum_{k=j-m+i}^{\min(j,d)} N_{i,j-k}\widehat{\psi}_k$$

IBNR part

$$\tilde{\mathsf{m}}_{ij}(\tilde{N}) = \sum_{k=\max(0,j-m+1)}^{\min(d,j-m+i-1)} \tilde{N}_{i,j-k}\widehat{\psi}_k$$

Bootstrap

Type of bootstrapping

- Non-parametric (residuals are resampled)
- Parametric

Parametric bootstrapping chosen for this model

- More natural choice – the model is based on specified underlying distributions

Error considered

- Only process error
- Both process and estimation errors

Variance of payments needed for the bootstrap procedure

- Estimated through the over-dispersion parameter φ
- Parameter φ estimated as in GLM

The over-dispersion parameter is suggested to be estimated using Pearson goodness-of-fit statistic

$$\widehat{\varphi} = \frac{1}{df} \sum_{i,j \in \mathcal{I}} \frac{\{X_{ij} - \widehat{\mathsf{m}}_{ij}(N)\}^2}{\widehat{\mathsf{m}}_{ij}(N)}$$

where

$$\widehat{\mathsf{m}}_{ij}(N) = \sum_{k=0}^{\min(j,d)} N_{i,j-k}\widehat{\psi}_k$$

and the degrees of freedom are

$$df = n - q$$

where *n* is dimension of *X*

$$n = m(m+1)/2$$

and *q* number of estimated delay parameters

$$q = d + 1$$

The estimator of the over-dispersion parameter can be naturally viewed as the estimator for

$$\varphi = \frac{1}{n} \sum_{i,j \in \mathcal{I}} \frac{\mathsf{v}_{ij}(N)}{\mathsf{m}_{ij}(N)}$$

where the variance is given by (recall the exact formula from the first model)

$$\mathsf{v}_{ij}(N) \quad = \quad \mathsf{Var}(X_{ij} \mid N) = \sum_{k=0}^{\min(j,d)} N_{i,j-k} \left\{ \sigma^2 p_k + \mu^2 p_k \left(1 - p_k\right) \right\}$$

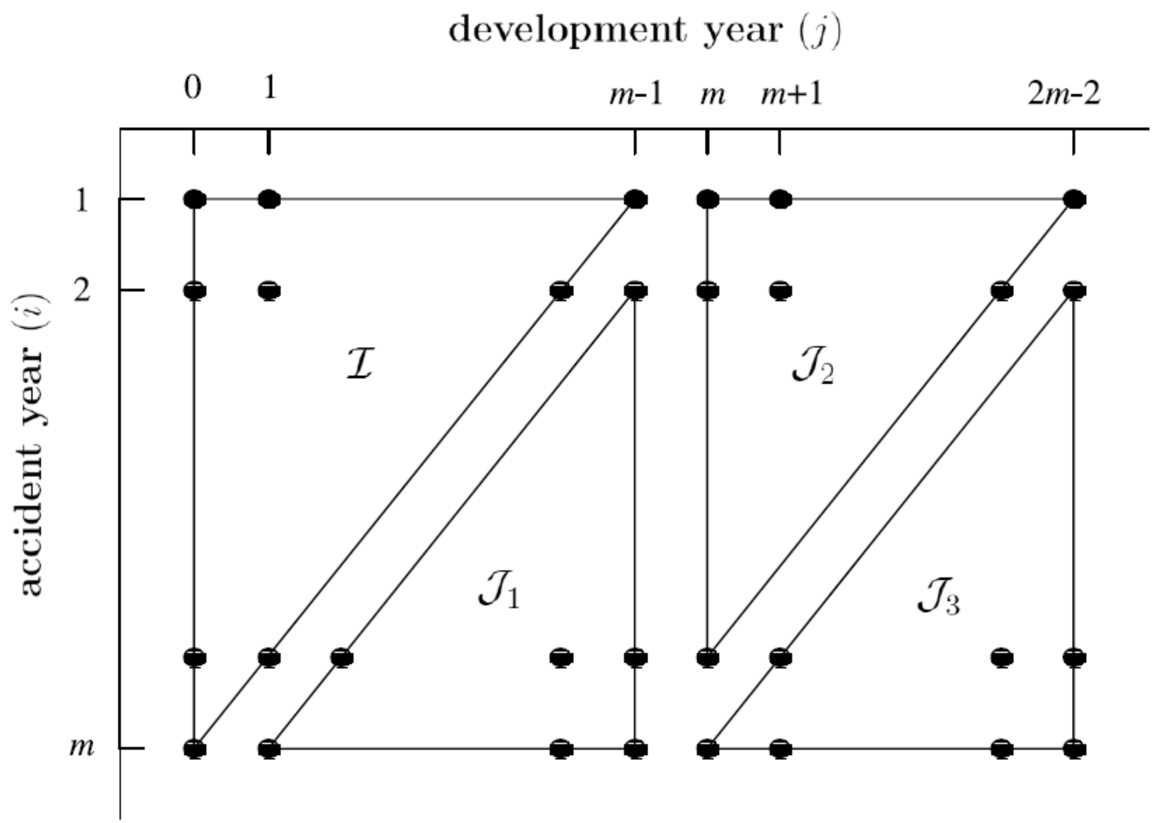
Using formulas for the mean and the variance, we get

$$\varphi = \frac{1}{n} \sum_{i,j \in \mathcal{I}} \frac{\mathsf{v}_{ij}(N)}{\mathsf{m}_{ij}(N)} = \frac{\sigma^2 + \mu^2}{\mu} - \frac{\mu}{n} \sum_{i,j \in \mathcal{I}} \frac{\sum_{k=0}^{\min(j,d)} N_{i,j-k} p_k^2}{\sum_{k=0}^{\min(j,d)} N_{i,j-k} p_k}$$

which implies the estimate for the parameter σ

$$\widehat{\sigma}^2 = \widehat{\mu}\widehat{\varphi} - \widehat{\mu}^2 + \frac{\widehat{\mu}^2}{n} \sum_{i,j \in \mathcal{I}} \frac{\sum_{k=0}^{\min(j,d)} N_{i,j-k} \widehat{p}_k^2}{\sum_{k=0}^{\min(j,d)} N_{i,j-k} \widehat{p}_k}$$

Triangles


development year (j)
0
1
m-1
m
m+1
2m-2
accident year (i)
1
2
m
$\mathcal{I}$
$\mathcal{J}_1$
$\mathcal{J}_2$
$\mathcal{J}_3$

2^{nd} version of the model

Parametric bootstrapping – underlying distributions (RBNS part)

RBNS part of the reserve

Incurred and reported counts: left-top triangle I

- Poisson distribution N_I

Aggregated claims X_{ij} arising from (already) incurred claims (triangles $I \cup J_1 \cup J_2$)

- Distribution $\mathbf{X}_{ij}(\boldsymbol{\theta}, N)$, where $\boldsymbol{\theta} = (\mathbf{p}, \mu, \sigma^2)$.
- It is constructed sequentially (let us remind the whole procedure)
  - Given incurred counts N_{ij}, number of payments N_{ijk}^{paid} are defined through the multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

  - The paid counts N_{ij}^{paid} are defined by

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

  - Individual claims distribution (severity distribution) may be chosen. We assumed only that $Y_{ij}(k)$ are i.i.d., independent of number of claims, independent of reporting and payment delay and then we derived the estimates for the mean μ and the variance σ^2.
    - Natural choice is gamma distribution with the mean μ and the variance σ^2, thus having density

$$f(y) = \frac{1}{\gamma(\lambda)\kappa^{\lambda}} y^{\lambda-1} \exp(-y/\kappa) \qquad \text{for } y > 0.$$

      with shape parameter $\lambda = \mu^2 / \sigma^2$ and scale parameter $\kappa = \sigma^2 / \mu$.

    - Given the count N_{ij}^{paid}, the aggregate claims X_{ij} are gamma distributed with shape $N_{ij}^{paid}\lambda$ and scale κ.

IBNR part of the reserve

Incurred but not yet reported counts: right-bottom triangle J_1

- Poisson distribution $N_{J1}(\omega)$

Aggregated claims X_{ij} arising from incurred but not yet reported claims

- Distribution $\mathbf{X}_{ij}(\theta, N_{J1})$
- Constructed analogically to the previous “RBNS case”

2nd version of the model

Bootstrapping – process variance + parameter estimation error

Process variance (stochastic error) only

- Simulation of unknown parts of the triangles (bottom-right + tail) from estimated parameters

Process variance and parameter estimation errors

- Estimated parameters used for simulation of new „left-top“ triangle(s)
- From these new triangles, „bootstrapped“ parameters are estimated
- From these „bootstrapped“ parameters, the unknown parts of triangles are simulated

Proposed algorithm for the bootstrapping procedure – RBNS part

Estimate of process variance only – do only steps 1, 4 and 5 (using parameters estimated in the step 1).

1. Parameters and distribution estimation

- Apply the procedure described for the best estimate to obtain estimates for p, μ, σ^2, λ, κ

2. Bootstrapping the data

- Keep the same counts N, but bootstrap the aggregate payments X* as follows
  - Simulate the delay (construct N_{ij}^{paid*} from given N_{ij} using the multinomial distribution estimated in the step 1)
  - Simulate the aggregate payments using gamma distribution with shape parameter $N_{ij}^{paid*}\lambda$ and scale parameter κ

3. Bootstrapping the parameters

- From the bootstrap data (N, X*) generated at step 2 obtain new estimates for p^*, μ^*, $\sigma 2^*$, λ^*, κ^*

4. Bootstrapping the RBNS prediction

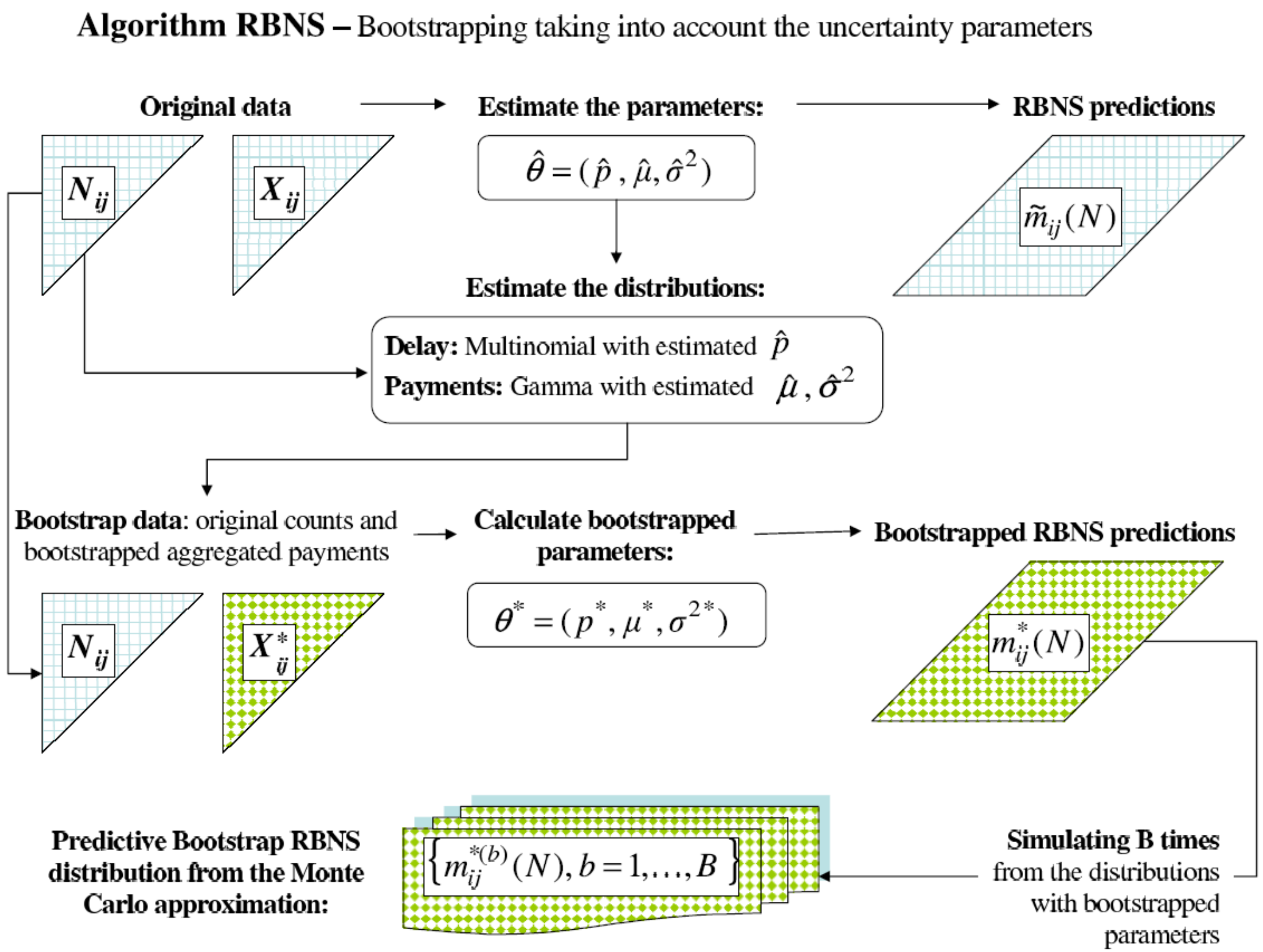
- Simulate the delay as in the step 2
- Simulate the aggregate payments as in the step 2
- Get the bootstrapped RBNS prediction

5. Monte Carlo approximation

- Repeat steps 2-4 B times and get the empirical bootstrap distribution of the RBNS part of the reserve

2nd version of the model

Bootstrapping – algorithm schema (RBNS part)



Proposed algorithm for the bootstrapping procedure – IBNR part

1. Parameters and distribution estimation

- Apply the procedure described for the best estimate to obtain estimates for p, μ, σ^2, λ, κ and use the chain-ladder to estimate future incurred claims counts (ω).

2. Bootstrapping the data

- Get new counts N^* and aggregate payments X^* as follows
  - Simulate new counts N^* (in the upper-left triangle) using Poisson distribution (with parameters estimated by the chain-ladder method in the step 1)
  - Using N^*, simulate X^* as in the second step of the RBNS procedure

3. Bootstrapping the parameters

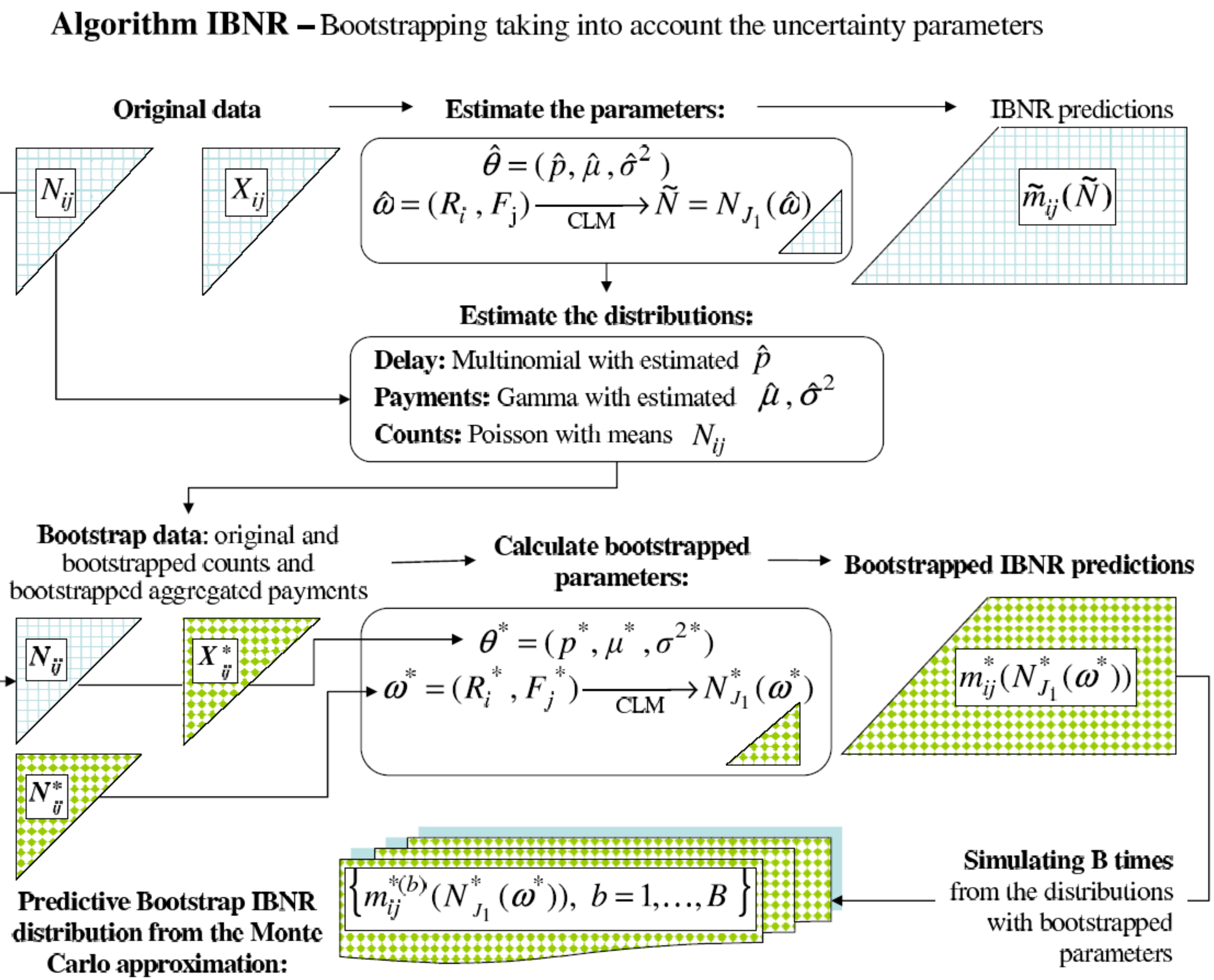
- From the bootstrap data (N^*, X^*) generated at step 2 obtain new estimates for p^*, μ^*, $\sigma 2^*$, λ^*, κ^* and use the chain-ladder to get bootstrapped future incurred claims counts.

4. Bootstrapping the RBNS prediction

- Simulate the delay for N_{ij}^* using p^*, i.e. construct $N_{ij}^{paid*, IBNR}$ analogously to the step 2 of the “RBNS” procedure
- Simulate the aggregate payments as in the step 2 and get the bootstrapped IBNR prediction (an. “RBNS” procedure)

5. Monte Carlo approximation

- Repeat steps 2-4 B times and get the empirical bootstrap distribution of the IBNR part of the reserve



Algorithm IBNR – Bootstrapping taking into account the uncertainty parameters
Original data
Estimate the parameters:
IBNR predictions
N_{ij}
X_{ij}
$\hat{\theta} = (\hat{p}, \hat{\mu}, \hat{\sigma}^2)$
$\hat{\omega} = (R_i, F_j) \xrightarrow{CLM} \tilde{N} = N_{J_1}(\hat{\omega})$
$\tilde{m}_{ij}(\tilde{N})$
Estimate the distributions:
Delay: Multinomial with estimated $\hat{p}$
Payments: Gamma with estimated $\hat{\mu}, \hat{\sigma}^2$
Counts: Poisson with means N_{ij}
Bootstrap data: original and bootstrapped counts and bootstrapped aggregated payments
Calculate bootstrapped parameters:
Bootstrapped IBNR predictions
N_{ij}
X^*_{ij}
N^*_{ij}
$\theta^* = (p^*, \mu^*, \sigma^{2*})$
$\omega^* = (R_i^*, F_j^*) \xrightarrow{CLM} N^*_{J_1}(\omega^*)$
$m^*_{ij}(N^*_{J_1}(\omega^*))$
Predictive Bootstrap IBNR distribution from the Monte Carlo approximation:
$\{m^{*(b)}_{ij}(N^*_{J_1}(\omega^*)),\ b = 1,\ldots,B\}$
Simulating B times from the distributions with bootstrapped parameters

Case study in the paper

- Adjustment for zero-claim is applied: $P(Y_{ij}^{(k)} = 0) = Q$, where Q set by expert judgment.
- Results – only best estimate available (MSEP, full distribution estimates etc. not considered in the paper)
  - Difference in the total best estimate is not large.
  - However, in the following paper it was suggested that using more data should imply less volatility (thus lower solvency requirement corresponding to VaR 99.5%)

i	IBNR	RBNS	TOTAL	CHAIN LADDER
2	628	605	1,233	1,685
3	1,350	4,514	5,863	29,379
4	1,510	43,623	45,133	60,638
5	1,967	94,526	96,493	101,158
6	2,579	171,633	174,212	173,802
7	3,168	299,136	302,304	249,349
8	5,349	509,334	514,684	475,992
9	14,280	852,144	866,423	763,919
10	254,499	1,135,678	1,390,177	1,459,860
Total	285,329	3,111,192	3,396,521	3,315,779

2nd version of the model

Case study on the same data – best estimate

Future	Calendar Year	RBNS	IBNR	RBNS+IBNR	CL
1	11	1307	93	1399	1354
2	12	720	78	798	754
3	13	494	34	529	489
4	14	323	26	349	318
5	15	188	20	208	185
6	16	117	12	130	115
7	17	65	9	74	63
8	18	37	5	42	36
9	19	0	6	6	2
10	20		1	1	
11	21		0.6	0.6	
12	22		0.4	0.4	
13	23		0.2	0.2	
14	24		0.1	0.1	
15	25		0.07	0.07	
16	26		0.04	0.04	
17	27		0.02	0.02	
18	28		0	0	
Total		3251	287	3538	3316

- England and Verrall (1999) bootstrap used for the chain-ladder
  - *Resampling Pearson residuals to obtain estimation error*
  - *Analytic adjustment for process error*

- See the difference between the mean in the table below and derived best estimates
  - *Too low number of simulations?*

	Bootstrap predictive distribution			
	RBNS	IBNR	Total	BCL
mean	3134	274	3408	3314
pe	327	60	340	345
1%	2464	148	2714	2588
5%	2646	183	2895	2780
50%	3105	272	3390	3287
95%	3722	378	4002	3911
99%	3987	435	4275	4061

pe = MSEP = mean square error of prediction

3^{rd} version of the proposed model

Double chain-ladder

Double chain-ladder

M.D.Martínez-Miranda, J. P. Nielsen, R. Verrall

Astin 2011, Conference paper

Main differences

- *Inflation parameter:* 1st and 2nd version did not allow for severity inflation
- *Parameter estimation* in the 3rd version uses only classical chain-ladder procedures applied twice on both considered triangles ➔ thus it is called *double chain-ladder*
- *3rd version, double chain-ladder,* can replicate the classical chain-ladder best-estimate
  - Thus the model can be viewed as another stochastic model for the classical chain-ladder method
- However, *double chain-ladder* provides not only a replica of the classical chain-ladder results but also three different sets of best estimates

Shared features with the 1st and 2nd version

- Cash-flow
- Split between “RBNS” and “IBNR” part
- Estimate of tail

1^{st} version of the model

Reminder: formal structure and assumptions

$\Delta_m = (X_{ij} : 1 \le i+j \le m)$ **traingle of claims paid**

$\aleph_m = (N_{ij} : 1 \le i+j \le m)$ **traingle of incurred claims**

- Claim is not usually paid immediately after notification. This motivates the introduction of the third triangle.

N_{ijk}^{paid} **– part of the** N_{ij} **claims fully paid with** k **periods delay after being reported,** $k = 0, \ldots, d$**;** d **is max. delay**

N_{ij}^{paid} **– number of claims incurred in period** i **and (fully) paid with** j **periods delay**

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

Assumptions

- N_{ij} independent, with over-dispersed Poisson distribution (ML estimate leads to classical CL algorithm)
- Given N_{ij}, the distribution of the numbers of paid claims follows a multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

- Claim settled with one payment (or as a zero claim). Thus, if we denote $Y_{ij}(k)$ the payment for the k-th claim incurred in period i settled with j periods delay, we have

$$X_{ij} = Y_{ij}(1) + Y_{ij}(2) + \ldots + Y_{ij}(N_{ij}^{paid})$$

- $Y_{ij}(k)$ i.i.d., independent of number of claims, independent of reporting and payment delay

$\Delta_m = (X_{ij} : 1 \leq i+j \leq m)$ **traingle of claims paid**

$\aleph_m = (N_{ij} : 1 \leq i+j \leq m)$ **traingle of incurred claims**

- Claim is not usually paid immediately after notification. This motivates the introduction of the third triangle.

N_{ijk}^{paid} **– part of the N_{ij} claims fully paid with k periods delay after being reported, $k = 0, \ldots, d$; d is max. delay**

N_{ij}^{paid} **– number of claims incurred in period i and (fully) paid with j periods delay**

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

Assumptions

- N_{ij} independent, with Poisson distribution (ML estimate leads to classical CL algorithm)
- Given N_{ij}, the distribution of the numbers of paid claims follows a multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

- Claim settled with one payment (or as a zero claim). Thus, if we denote $Y_{ij}(k)$ the payment for the k-th claim incurred in period i settled with j periods delay, we have

$$X_{ij} = Y_{ij}(1) + Y_{ij}(2) + \ldots + Y_{ij}(N_{ij}^{paid})$$

- $Y_{ij}(k)$ are mutually independent with distributions f_i. Further, for the mean μ_i and the variance σ_i^2, we assume that

$$\mu_i = \mu\gamma_i \text{ and } \sigma_i^2 = \sigma^2\gamma_i$$

The derivation proceeds in a way very similar to the 1st version of the model.

For the conditional mean and variance, we have

$$\mathrm{E}[X_{ij}|\aleph_m] = \mathrm{E}[N_{ij}^{paid}|\aleph_m]\mathrm{E}[Y_{ij}^{(k)}] = \sum_{l=0}^{\min(j,d)} N_{i,j-l}p_l\mu\gamma_i$$

$$\begin{aligned}\mathrm{V}[X_{ij}|\aleph_m] &\approx \frac{\sigma_i^2+\mu_i^2}{\mu_i}\mathrm{E}[X_{ij}|\aleph_m] \\ &= \gamma_i\frac{\sigma^2+\mu^2}{\mu}\mathrm{E}[X_{ij}|\aleph_m] \\ &= \varphi_i\mathrm{E}[X_{ij}|\aleph_m].\end{aligned}$$

Thus, an over-dispersed Poisson model can again be used…

- Construct (psuedo log-)likelihood function
- Maximization gives ML estimate of parameters, over-dispersion can be then estimated using Pearson g.o.f. statistic

…but, as in the 2nd version of the model, an alternative analytical approach is suggested

Double chain-ladder

Derivation (cont'd)

Again, put

$$d = m - 1$$

We substitute the probabilities p_i which have a natural constraint

$$p_1 + p_2 + \ldots + p_d = 1$$

with parameters π_i without this constraint. That is, we have a conditional mean

$$\mathrm{E}[X_{ij}|\aleph_m] = \sum_{l=0}^{j} N_{i,j-l}\pi_l \mu \gamma_i$$

From the classical chain-ladder method (with classical Mack identification), we obtain parameters, so that

$$\mathrm{E}[N_{ij}] = \alpha_i \beta_j$$

Thus, for the unconditional mean, we have

$$\mathrm{E}[X_{ij}] = \alpha_i \mu \gamma_i \sum_{l=0}^{j} \beta_{j-l}\pi_l$$

However, we can estimate $\mathrm{E}[X_{ij}]$ by the chain-ladder method again applied on the triangle of paid claims.

Using CL method on the triangle of paid claims, we get parameters, so that it is satisfied

$$\mathrm{E}[X_{ij}] = \widetilde{\alpha}_i \widetilde{\beta}_j$$

A direct comparison with the previous formula

$$\mathrm{E}[X_{ij}] = \alpha_i \mu \gamma_i \sum_{l=0}^{j} \beta_{j-l} \pi_l$$

leads to a natural identification

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i$$

$$\sum_{l=0}^{j} \beta_{j-l} \pi_l = \widetilde{\beta}_j$$

Using this identification of parameters:

1. **will replicate the chain-ladder results in the framework of DCL method (if tail is ignored);**
2. **provides a natural way to estimate parameters necessary for DCL analytically.**

Double chain-ladder

Derivation (cont'd)

The second identification formula

$$\sum_{l=0}^{j} \beta_{j-l}\pi_l = \widetilde{\beta}_j$$

allows to estimate π_l since β_j and $\widetilde{\beta_j}$ are estimated by the chain-ladder algorithm applied on the triangles of incurred counts and paid claims respectively.

For the estimate of π_l , one needs to solve a linear system

$$\begin{pmatrix} \widetilde{\beta}_0 \\ \vdots \\ \vdots \\ \widetilde{\beta}_{m-1} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 & \cdots & 0 \\ \beta_1 & \beta_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \beta_{m-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}$$

From the estimate of π_l , one can estimate p_l by several ways, authors suggested a very simple method

- Maximal delay *d* is estimated by summing the number of succesive estimates of π_l until a number greater or equal to one is achieved. Then *d* is equal to the count of summands and it is put

$$\widehat{p}_l = \widehat{\pi}_l, l = 0, \dots, d-1,$$
$$\widehat{p}_d = 1 - \sum_{l=0}^{d-1} \widehat{p}_l.$$

- In practice, there should be (!) little difference between π_l and p_l .

Double chain-ladder

Derivation (cont'd)

Other parameters can be estimated using the first identification formula

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i$$

by

$$\widehat{\gamma}_i = \frac{\widehat{\widetilde{\alpha}}_i}{\widehat{\alpha}_i \widehat{\mu}}$$

The model is technically overparametrised, but it is natural to put $\gamma_1 = 1$ and estimate

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}$$

This gives us the *Double-chain ladder* predictor

$$\widehat{X}_{ij}^{DCL} = \sum_{l=0}^{\min(j,d)} N_{i,j-l} \widehat{p}_l \widehat{\mu} \widehat{\gamma}_i$$

Double chain-ladder

Derivation (cont'd)

Finally, we can estimate the over-dispersion parameter using

$$\widehat{\varphi} = \frac{1}{n-(d+1)} \sum_{i,j \in \mathcal{I}} \frac{(X_{ij} - \widehat{X}_{ij}^{DCL})^2}{\widehat{X}_{ij}^{DCL} \widehat{\gamma}_i}$$

Where

$$n = m(m+1)/2$$

and

The variance factors are then estimated by

$$\widehat{\sigma}^2 = \widehat{\mu}\widehat{\varphi} - \widehat{\mu}^2$$

$$\widehat{\sigma}_i^2 = \widehat{\sigma}^2 \widehat{\gamma}_i^2$$

The DCL method offers four basic options for the best estimate of provisions for claims outstanding

1. Using π_l parameters and actual incurred counts (in the left-top triangle where it is possible)

$$\widehat{X}_{ij}^{rbns(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i$$

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i.$$

2. Using π_l parameters and CL predictions in the whole square. **This option replicates the CL results applied on the triangle of claims paid.**

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i$$

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i.$$

3. and 4. Replacing π_l set of parameters by the p_l set.

Note that the tail can also be estimated using

$$\sum\nolimits_{(i,j)\in\mathcal{J}_2\cup\mathcal{J}_3} \sum_{l=0}^{\min(j,d)} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i$$

Again, π_l set of parameters can be replaced by the p_l set.

Bootstrap procedure can be applied without any significant change compared to the 2nd model.

Double chain-ladder

Case study (same data) – best estimate

	DCL			MNNV			
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	CL
1	1260	97	1357	1307	93	1399	1354
2	672	83	754	720	78	798	754
3	453	35	489	494	34	529	489
4	292	26	319	323	26	349	318
5	165	20	185	188	20	208	185
6	103	12	115	117	12	130	115
7	54	9	63	65	9	74	63
8	30	5	36	37	5	42	36
9	0	5	5	0	6	6	2
10	1		1		1	1	
11	0.6		0.6		0.6	0.6	
12	0.4		0.4		0.4	0.4	
13	0.2		0.2		0.2	0.2	
14	0.1		0.1		0.1	0.1	
15	0.06		0.06		0.07	0.07	
16	0.03		0.03		0.04	0.04	
17	0.01		0.01		0.02	0.02	
Total	3030	296	3326	3251	287	3538	3316

Double chain-ladder

Case study (same data) – bootstrap

	Bootstrap predictive distribution						
	DCL			MNNV			
	RBNS	IBNR	Total	RBNS	IBNR	Total	CL
mean	3013	294	3307	3134	274	3408	3314
pe	279	52	300	327	60	340	345
1%	2415	198	2661	2464	148	2714	2588
5%	2575	215	2821	2646	183	2895	2780
50%	2995	289	3291	3105	272	3390	3287
95%	3505	389	3813	3722	378	4002	3911
99%	3649	425	4020	3987	435	4275	4061

Questions

&

Comments

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Thank you

Petr Pošta

cutting through complexity