Infinitely Stochastic Micro Reserving

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Infinitely stochastic micro reserving

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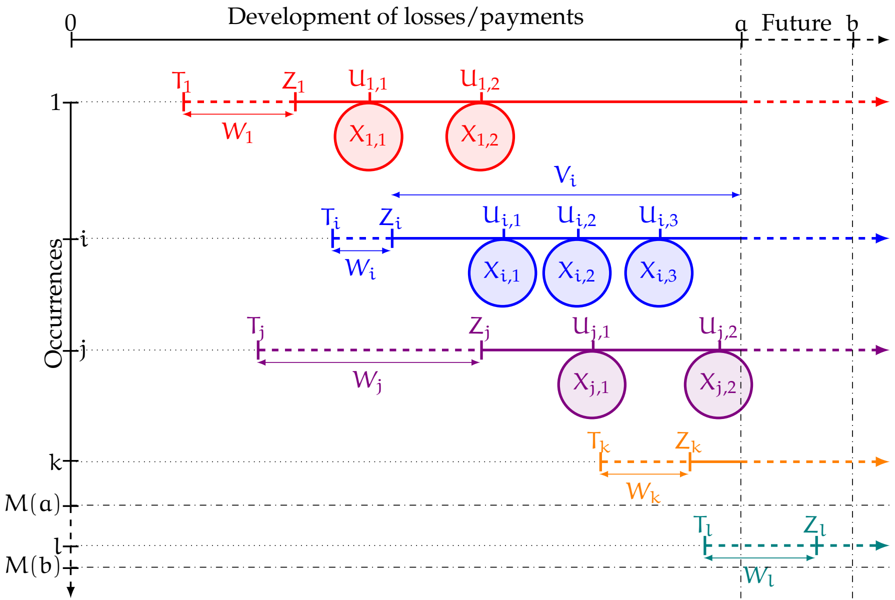
Motivation

Forecasting costs … a front burner in empirical economics

- **Stochastic prediction of future expenses** based on the individual (micro) developments of recorded events
- Consider a firm, enterprise, institution, or state, which possesses knowledge about particular **historical events**
- For each event, there is a series of several **related subevents**: payments or losses spread over time
- Nevertheless, the **issue** is that some already occurred events do not have to be necessarily reported
- The aim lies in **forecasting future subevent flows** coming from already reported, occurred but not reported, and yet not occurred events

Illustration



Setup

The event's **lifetime** can be described as follows:

- The ith loss occurs at the **occurrence time** T_i
- Such a loss is often reported (e.g., to a financial company) not immediately after the event, but after the **reporting delay** W_i, which is the time difference between the **occurrence** and **observation epoch**
- $Z_i = T_i + W_i$ stands for the ith **reporting/notification time**
- Our **observation history** for the reported losses is a time interval $[0, a]$
- The amount of the kth **payment** within the ith loss paid at time $U_{i,k}$ is represented by $X_{i,k}$

Stochastic Framework – Recap

Non-homogeneous **Hawkes** process $\{M(t)\}_{t \geqslant 0}$

$\equiv$ non-stationary (time-varying) self-exciting point process

$:=$ a simple point process such that

$$\mathsf{P}[\triangle M(t) = 1 | M(s)(s \leqslant t)] = \psi(t)\triangle t + o(\triangle t)$$
$$\mathsf{P}[\triangle M(t) > 1 | M(s)(s \leqslant t)] = o(\triangle t)$$

where

$$\psi(t) = \mu(t) + \int_{[0,t)} \varkappa(t-u) dM(u)$$

- a time-varying background/baseline intensity $\mu(t) > 0$ with $\varkappa \equiv 0$ represents a non-homogeneous Poisson process
- an excitation function $\varkappa$ measures the influence of an event on the intensity process introducing the dependence between the occurring events

Stochastic Framework – Assumptions on M

The time ordered reporting times $\{Z_i\}_{i\in\mathbb{N}}$ are arrival times of a Hawkes process $\{M(t)\}_{t\geqslant 0}$, i.e., $M(t) = \sum_{i=1}^{\infty} \mathbb{1}\{Z_i \leqslant t\}$, with a parametric intensity

$$\psi(t;\boldsymbol{\rho}) = \mu(t;\boldsymbol{\rho}) + \sum_{i:\, Z_i<t} \varkappa(t - Z_i;\boldsymbol{\rho})$$

such that $\mu(\cdot;\boldsymbol{\rho}) > 0$ and $\varkappa(\cdot;\boldsymbol{\rho}) \geqslant 0$ are continuous, $\boldsymbol{\rho} \in \boldsymbol{R} \subseteq \mathbb{R}^q$, and $\boldsymbol{R}$ is an open convex set.

Stochastic Framework – Assumptions on N_i

The ordered payment times $\{U_{i,1}, U_{i,2}, \ldots\}$ of the ith loss are arrival times of a Hawkes process $\{N_i(t)\}_{t\geqslant 0}$, i.e., $N_i(t) = \sum_{k=1}^{\infty} \mathbb{1}\{U_{i,k} \leqslant t\}$. The processes $\{N_i(t)\}_{t\geqslant 0}$, $i = 1, 2, \ldots$ are independent having parametric intensities

$$\lambda(t, Z_i; \boldsymbol{\theta}) = \xi(t, Z_i; \boldsymbol{\theta}) + \sum_{j\colon U_{i,j}<t} \zeta(t - U_{i,j}, Z_i; \boldsymbol{\theta})$$

such that $\zeta(\cdot, z; \boldsymbol{\theta}) \geqslant 0$ is continuous, $t \mapsto \xi(t, z; \boldsymbol{\theta})$ is positive and continuous for $t > z$, and $\xi(t, z; \boldsymbol{\theta}) = 0$ for $t \leqslant z$, where $z > 0$, $\boldsymbol{\theta} \in \boldsymbol{P} \subseteq \mathbb{R}^p$, and $\boldsymbol{P}$ is an open convex set.

Applications

The proposed class of models—**infinitely stochastic processes**—is a very rich and general class that nests, for examples, doubly stochastic (Cox) processes

- **Operational risk** of banks covering fraud, system failures, security, privacy protection, terrorism, legal risks, employee compensation claims, physical (e.g., infrastructure shutdown) or environmental risks
- **War damages**, **epidemics**, **drug prescription**, **startups**
- **Advertising and commercials**, **digital payments** (Apple Pay)
- We concentrate in more details on the **actuarial claims reserving** task and **exemplify the proposed methodology** through analyses of two insurance lines of business

Data

We observe a collection

$$\left\{T_i, Z_i, \{U_{i,j}, X_{i,j}\}_{j=1,\dots,N_i(a)}\right\}_{i=1,\dots,M(a)}$$

- Panels of count processes $\{N_i(t)\}_{t>0}$ for $i = 1, \dots, M(t)$ that can be represented as $\{\boldsymbol{N}(t)\}_{t>0}$
- Thus, the claim notifications together with the claim payments can be viewed as a **marked Hawkes process with other Hawkes processes as marks**

$$\left\{\{M(t)\}_{t\geqslant 0}, \{\boldsymbol{N}(t)\}_{t\geqslant 0}\right\}$$

Maximum Likelihood

The log-likelihood function $\ell(t;\boldsymbol{\theta}) = -\sum_{i=1}^{M(t)} g_i(t;\boldsymbol{\theta})$, where

$$g_i(t;\boldsymbol{\theta}) := \int_0^t \xi(z, Z_i;\boldsymbol{\theta})dz - \sum_{k=1}^{N_i(t)} \left[\int_{U_{i,k}}^t \zeta(z - U_{i,k}, Z_i;\boldsymbol{\theta})dz \right.$$

$$\left. - \log\left\{ \xi(U_{i,k}, Z_i;\boldsymbol{\theta}) + \sum_{j:\, U_{i,j}<U_{i,k}} \zeta(U_{i,k} - U_{i,j}, Z_i;\boldsymbol{\theta}) \right\} \right]$$

The ML estimator of the true unknown $\boldsymbol{\theta}_0$ is

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{P}} \sum_{i=1}^{M(t)} g_i(t;\boldsymbol{\theta})$$

Consistency (and Asymptotic Normality)

Theorem

Under some regularity assumptions,

$$\mathcal{J}^{1/2}(t,\boldsymbol{\theta}_0)\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right) = -\mathcal{J}^{-1/2}(t,\boldsymbol{\theta}_0)\sum_{i=1}^{M(t)} \partial_{\boldsymbol{\theta}_0} g_i\,(t;\boldsymbol{\theta}) + o_{\mathsf{P}}(1), \quad t\to\infty,$$

where

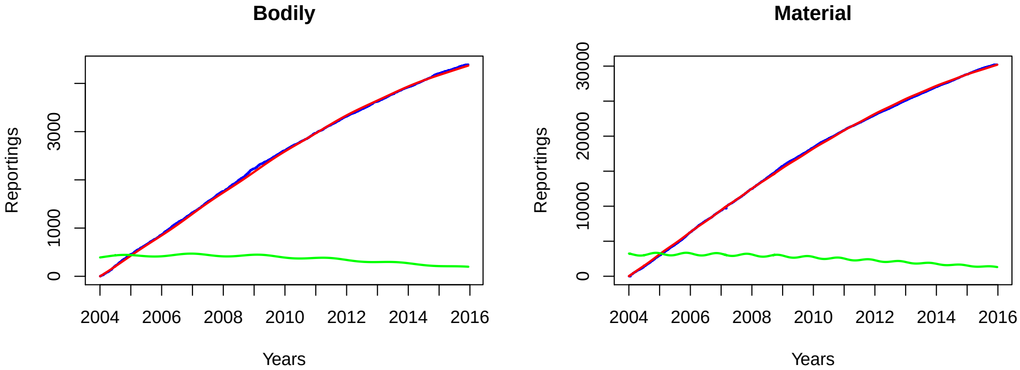
$$\mathcal{J}(t,\boldsymbol{\theta}_0) := \mathsf{E}\sum_{i=1}^{M(t)}\int_{Z_i}^{t}\frac{\{\partial_{\boldsymbol{\theta}_0}\lambda(\tau,Z_i;\boldsymbol{\theta})\}^{\otimes 2}}{\lambda(\tau,Z_i;\boldsymbol{\theta}_0)}d\tau.$$

Umbrella Model

Parametric approach:

- The **reporting times** Z_i's follow a non-homogeneous Hawkes process with a parametric intensity function
- The **reporting delays** W_i's follows a time-varying continuous parametric distribution conditional on the reporting times ... $f_W\{\cdot; w(Z_i, \boldsymbol{\vartheta})\}$
- The **payment times** for each claim i are represented by arrival times of a non-homogeneous Hawkes process $N_i(t)$, where N_i is a mark of M
- The **payment amounts** $X_{i,j}$'s are modeled similarly to the reporting delays via a time-varying parametric conditional distribution ... $f_X\{\cdot; v(Z_i, \boldsymbol{\sigma})\}$

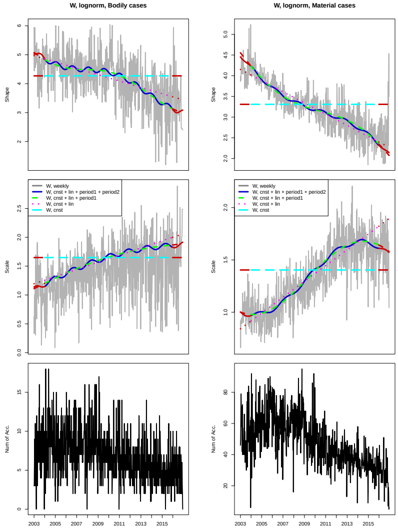
M(t) as Hawkes – Number of Reported Claims



Empirical (observed – blue) and estimated (red) cumulative intensity; estimated intensity (green) – prediction uses only data up to the end of 2015);

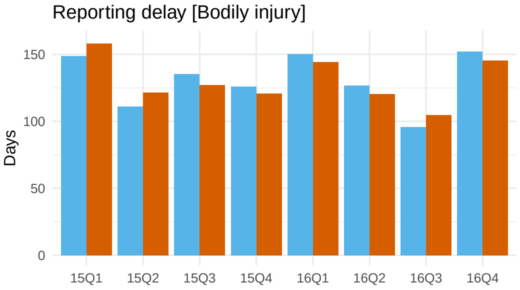
The linear Volterra integral equation $\bar{\mu}(t;\boldsymbol{\rho}) = \mu(t;\boldsymbol{\rho}) + \int_0^t \varkappa(t-z;\boldsymbol{\rho})\bar{\mu}(z;\boldsymbol{\rho})dz$ for the mean intensity $\bar{\mu}(t;\boldsymbol{\rho}) = \mathsf{E}[dM(t)]/dt$

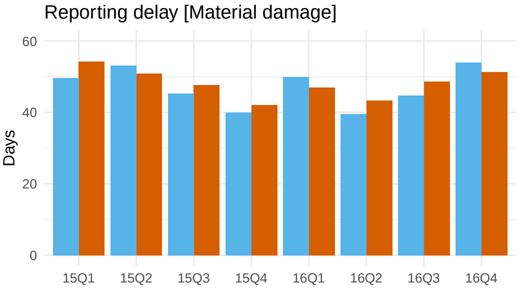
Reporting Delay W_i



Weekly estimates (solid grey) and conditional temporal models: constant (cyan dashed), linear trend (pink dotted), linear trend and one period (green dashed), and linear trend with two periods (blue solid) for shape (top panels) and scale (middle panels) of the log-normal distribution of the reporting delay W_i. The extrapolated periods are depicted in red and the numbers of accidents are in black

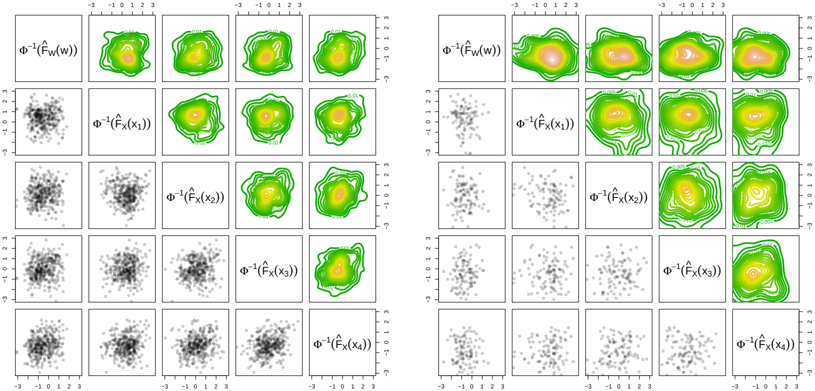
Reporting Delay W_i – Diagnostics





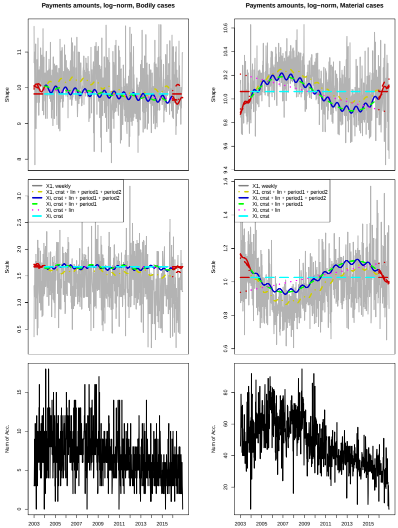
Quarterly averaged (with respect to the reporting time) reporting (waiting) delays in days—observed in blue and predicted in orange for the period 2015–2016 (prediction uses only data up to the end of 2015)

Dependencies



Pairwise relationship between the reporting delay and the claim payment amounts (bodily injury claims—left; material damage claims—right)

Payment Amounts $X_{i,j}$



Weekly estimates (only $X_{1,t}$ in solid grey) and conditional temporal models: constant (cyan dashed), linear trend (pink dotted), linear trend and one period (green dashed), and linear trend with two periods (blue solid for $X_{i,t}$ and yellow dot-dashed for $X_{1,t}$) for shape (top panel) and scale (middle panel) of the log-normal distribution of the $X_{i,t}$. The extrapolated periods are depicted in red and the numbers of accidents are in black

Loss Reserves

- **One year** ahead prediction, although the **ultimate** prediction is feasible
- Three types of reserves:
  - RBNS (Reported But Not Settled) … $T_i < Z_i \leqslant a$;
  - IBNR (Incurred But Not Reported) … $T_i \leqslant a < Z_i$;
  - UPR (Unearned Premium Reserve) … $a < T_i < Z_i$.
- One can predict **separately** the RBNS, IBNR, and UPR reserves

Prediction of Distribution of the Future Flows

Input: Collection of observations $\left\{T_i, Z_i, \{U_{i,j}\}_{j=1}^{N_i(a)}, \{X_{i,j}\}_{j=1}^{N_i(a)}\right\}_{i=1}^{M(a)}$ and number of Monte Carlo simulation's runs S

Output: Simulated predictive distribution of the total future payments in $(a, b]$, i.e., the empirical distribution where probability mass $1/S$ concentrates at each of ${}_{(1)}P(a, b), \ldots, {}_{(S)}P(a, b)$

1: MLE for the parametric intensity of the reporting times
2: MLE for the parametric densities of the reporting delays
3: MLE for the parametric intensities of the payments times
4: MLE for the parametric densities of the payment amounts

Monte Carlo the Future Time Window $(a, b]$

for $s = 1$ to S **do** // to obtain the empirical distribution
  generate a realization of $\{{}_{(s)}M(t)\}_{t \geqslant 0}$ as the arrival times $\{{}_{(s)}Z_{M(a)+1}, \ldots, {}_{(s)}Z_{{}_{(s)}M(b)}\}$
  for $i = 1$ to $M(a)$ **do** // already reported claims
    generate a realization of $\{{}_{(s)}N_i(t)\}_{t \geqslant 0}$ as the arrival times $\{{}_{(s)}U_{i,N_i(a)+1}, \ldots, {}_{(s)}U_{i,{}_{(s)}N_i(b)}\}$
    generate the payment amounts $\{{}_{(s)}X_{i,N_i(a)+1}, \ldots, {}_{(s)}X_{i,{}_{(s)}N_i(b)}\}$
  end for
  for $i = M(a) + 1$ to ${}_{(s)}M(b)$ **do** // new reported claims
    generate a realization of $\{{}_{(s)}N_i(t)\}_{t \geqslant 0}$ as the arrival times $\{{}_{(s)}U_{i,N_i(a)+1}, \ldots, {}_{(s)}U_{i,{}_{(s)}N_i(b)}\}$
    generate the payment amounts $\{{}_{(s)}X_{i,N_i(a)+1}, \ldots, {}_{(s)}X_{i,{}_{(s)}N_i(b)}\}$
    generate the reporting delays $\{{}_{(s)}W_{N_i(a)+1}, \ldots, {}_{(s)}W_{{}_{(s)}N_i(b)}\}$
  end for

Claims Reserves

for $s = 1$ to S **do** // to obtain the empirical distribution
    total future payments ${}_{(s)}P(a,b) = \sum_{i=1}^{{}_{(s)}M(b)} \sum_{j=N_i(a)+1}^{{}_{(s)}N_i(b)} {}_{(s)}X_{i,j}$
    get ${}_{(s)}RBNS(a,b) = \sum_{i=1}^{M(a)} \sum_{j=N_i(a)+1}^{{}_{(s)}N_i(b)} {}_{(s)}X_{i,j}$
    get ${}_{(s)}IBNR(a,b) = \sum_{i=M(a)+1}^{{}_{(s)}M(b)} \left[\sum_{j=1}^{{}_{(s)}N_i(b)} {}_{(s)}X_{i,j} \right] \mathbb{1}\{{}_{(s)}Z_i - {}_{(s)}W_i \leqslant a\}$
    get ${}_{(s)}UPR(a,b) = \sum_{i=M(a)+1}^{{}_{(s)}M(b)} \left[\sum_{j=1}^{{}_{(s)}N_i(b)} {}_{(s)}X_{i,j} \right] \mathbb{1}\{{}_{(s)}Z_i - {}_{(s)}W_i > a\}$
end for

Quantitative Behavior of the Predicted Reserves

- The reserves $R := RBNS(a, b) + IBNR(a, b)$
- The prediction of R relying on Wald's identity

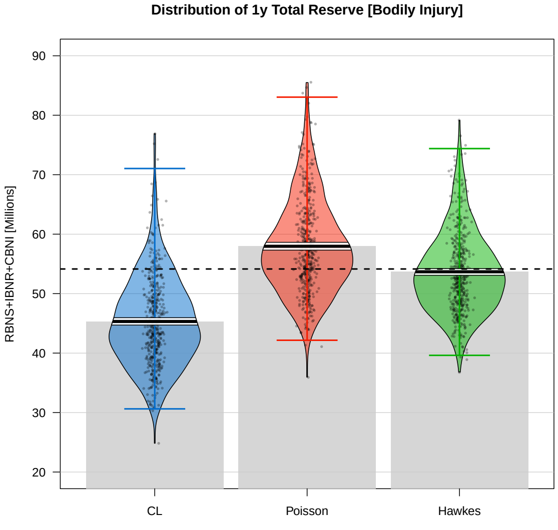
$$\widehat{R} = \sum_{i=1}^{M(a)} \sum_{j=N_i(a)+1}^{\widehat{E}N_i(b)} \widehat{E}X_{i,j} + \sum_{i=M(a)+1}^{\widehat{E}M(b)} \left(\sum_{j=1}^{\widehat{E}N_i(b)} \widehat{E}X_{i,j} \right) \widehat{P}[Z_i - W_i \leqslant a]$$

- The estimated expectations $\widehat{E}$ and probabilities $\widehat{P}$ are obtained from their theoretical counterparts by plugging-in the estimated parameters of densities and intensities
- The within-sample mean square error of prediction (MSEP) of R comes from the variance-bias decomposition $MSEP(R) = \mathsf{Var}\, R + bias^2(R)$
- The MSEP estimate becomes $\widehat{MSEP}(R) = \widehat{\mathsf{Var}} R + \widehat{bias}^2(R)$ such that $\widehat{\mathsf{Var}} R = \frac{1}{S} \sum_{s=1}^{S} \left[{}_{(s)}R - \frac{1}{S} \sum_{r=1}^{S} {}_{(r)}R \right]^2$ and $\widehat{bias}^2(R) = \left(\frac{1}{S} \sum_{s=1}^{S} {}_{(s)}R - \widehat{R} \right)^2$

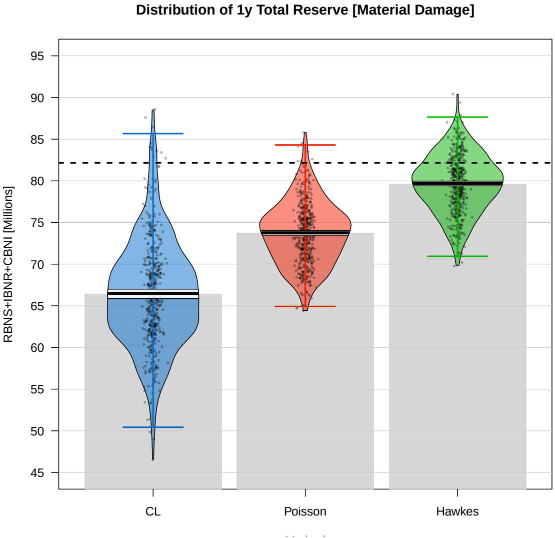
Reserving Methods – Comparison

- The **bootstrap chain-ladder** (England and Verrall, 1999), traditional 'macro' standard actuarial technique, in combination with linear extrapolation of the reported claims for the next year – denoted by **CL**
- A **micro reserving** technique theoretically proposed by Norberg (1993) and practically implemented by Antonio and Plat (2014) with some adjustments – **Poisson**
- A micro method of marked non-homogeneous Hawkes process with other non-homogeneous Hawkes processes as marks (Maciak, Okhrin, and Pešta, 2021) – **Hawkes**

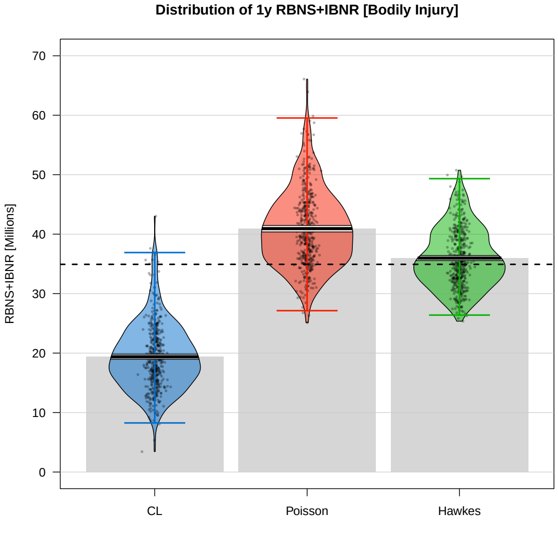
Bodily Injury (Total Reserves)



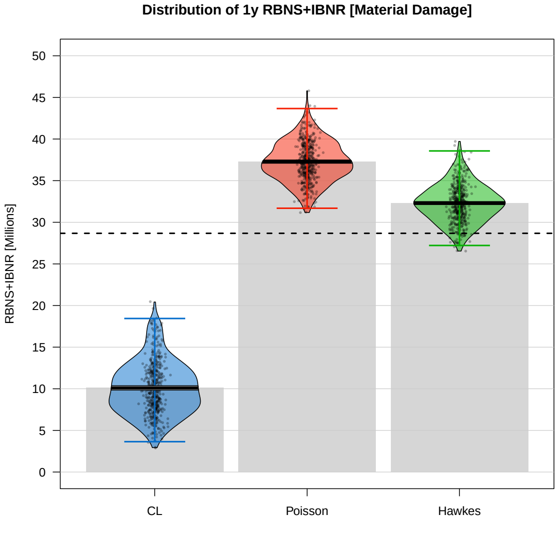
Material Damage (Total Reserves)



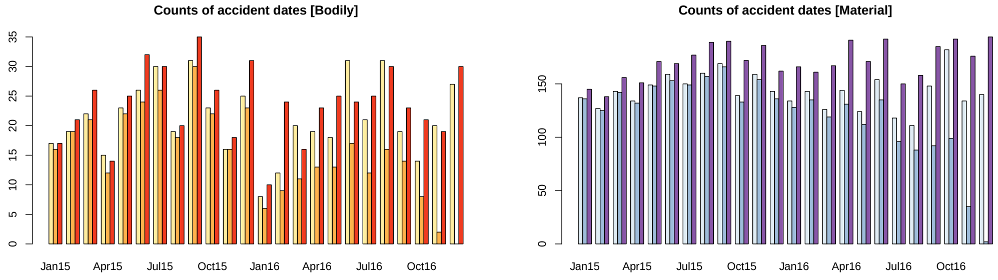
Bodily Injury (RBNS+IBNR)



Material Damage (RBNS+IBNR)



Back fitted – Predicted truncated accident times



Triplets of bars represent: observed counts of the accident times from the database up to the end of 2016 (in yellow/light blue); observed counts of the accident times from the database up to the end of 2015 (in orange/darker blue); and predicted counts of the accident times based on the data till the end of 2015 (in red/violet)

Conclusions

Infinitely stochastic **micro forecasting**:

- A stochastic prediction method for future losses/costs relying on the **individual developments** of the recorded historical events
- Quantifying reserving risk in non-life insurance inadvertently yields to a theoretical framework of the **marked non-homogeneous Hawkes process with other non-homogeneous Hawkes processes as marks**
- A proper statistical **inference** relying on simple and verifiable assumptions is derived for **infinitely stochastic Hawkes processes**
- **Utility for solvency** of the insurance company … to model the probabilistic behavior of the future losses' occurrences, the occurrences of the incurred but not reported losses, the lengths of the reporting delays, and the frequency and severity of the loss payments in time

Thank you for your attention !

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