KPMG

cutting through complexity

Double chain-ladder and its extensions

Seminář z aktuárských věd

14 November 2014

Contents

Reserving methods “in practice” based on triangles

- Chain-ladder
  - Triangle of paid claims
  - Triangle of incurred claims
  - Triangle of reported claims
  - Triangle of incurred counts
- Münich chain-ladder
  - Triangle of paid claims + Triangle of incurred claims

Goals

- Best estimate
- Mean square error of prediction
- VaR 99.5%
- Full distribution
  - Fit a chosen distribution to the first two moments
  - Bootstrap (non-parametric / parametric)

Aggregation of data

Triangles: aggregated data

+ Convenient presentation

– Loss of information which in some cases may lead to a poor performance

Individual claims modeling

+ No loss of information

– Usually complex models with lots of parameters

– Require large datasets (which might not be available)

– Might be computationally expensive

Trade-off

Using simple model vs. Using all information

Double chain-ladder

- „Triangular method“ based on micro-level assumptions
- Using more information (two triangles + possibly additional information)
  - Key question for this presentation: Does it lead necessarily to better performance?

Double chain-ladder

First moment formulation

Double chain-ladder

M.D. Martínez-Miranda, J. P. Nielsen, R. Verall

Astin Bulletin 2012

(published version of the paper)

Chain-ladder

- Using one triangle (paid / incurred / reported)
- All sources of delay (reporting, payment) incorporated in one development pattern

Proposed alternative

- Basic idea is to separate the sources of delay ➔ using more than one triangle
  - Triangle of incurred counts → reporting delay
  - Triangle of claims paid → payment delay
- It naturally leads to frequency-severity model

- *Using triangle of incurred claims as a further supplementary source of information considered in BDCL model*

Chain-ladder

- First, there was an algorithm without an underlying stochastic model
- Underlying stochastic models added later
  - Poisson model (CL is maximum-likelihood estimator)
  - **Mack distribution-free model**
  - …

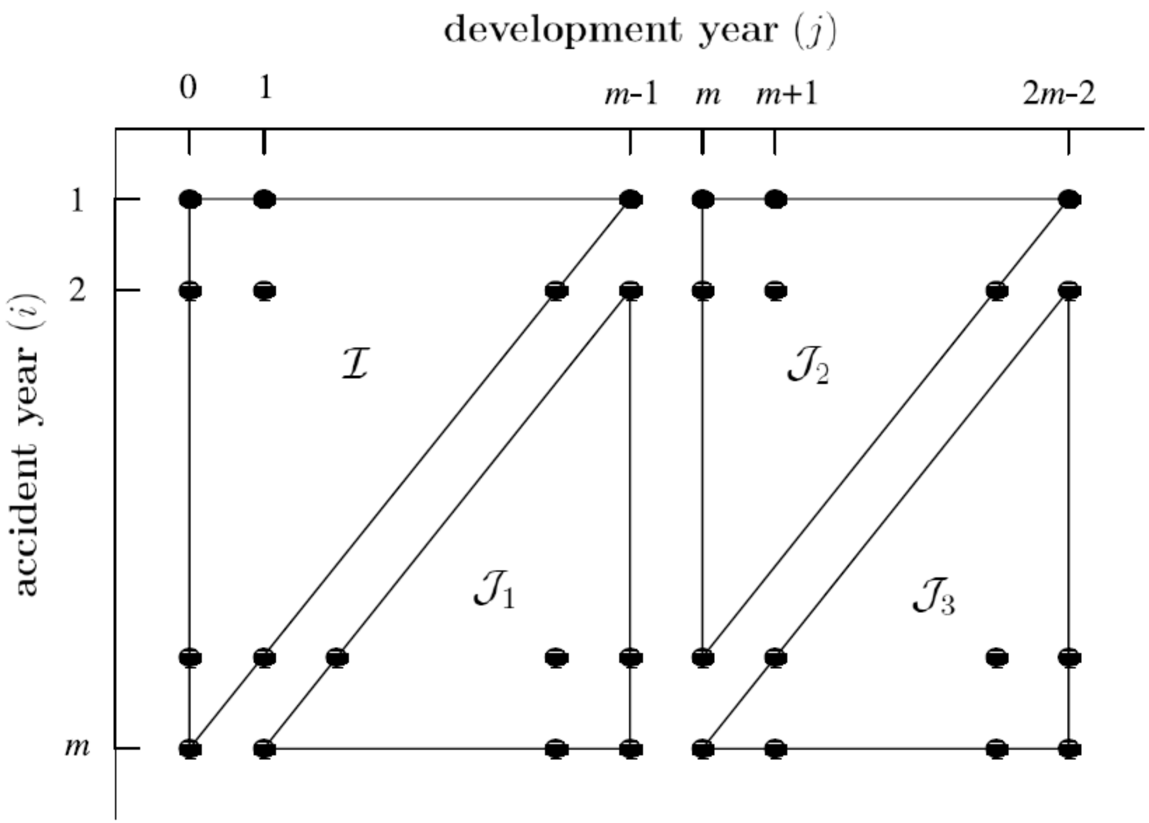
Double chain-ladder

- First, there was an underlying exact compound Poisson model based on more detailed data
- Model to be used in practice – *double chain-ladder* – was originally derived as its approximation
  - “Best estimate“ algorithm consists of using ordinary chain-ladder twice
  - “Distribution-free“ formulation for the best estimate proposed later
  - For VaR calculations, parametric model is recommended

New features compared to ordinary chain-ladder applied to the triangle of claims paid

- Provides separate estimates of future cash-flows from reported claims (RBNS) and not yet reported claims (IBNR)
- Provides a “consistent estimate of tail“

Double chain-ladder
Triangles – notation



Double chain-ladder

Formal structure and assumptions – original parametric model

$\mathcal{\Delta}_m = \{X_{ij} : (i,j) \in \mathcal{I}\}$ traingle of claims paid

$\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}\}$ traingle of incurred claims counts

- Claim is not usually paid immediately after notification. This motivates the introduction of the third triangle.

N_{ijk}^{paid} – part of the N_{ij} claims fully paid with k periods delay after being reported, $k = 0, \ldots, d$; d is max. delay

N_{ij}^{paid} – number of claims incurred in period i and (fully) paid with j periods delay

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

Assumptions

- N_{ij} independent, with Poisson distribution (ML estimate leads to classical CL algorithm)
- Given N_{ij}, the number of payments follows a multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \mathrm{Multi}(N_{ij}; p_0, \ldots, p_d)$$

- Claim settled with one payment (or as a zero claim). Thus, if we denote $Y_{ij}(k)$ the payment for the k-th claim incurred in period i settled with j periods delay, we have

$$X_{ij} = Y_{ij}(1) + Y_{ij}(2) + \ldots + Y_{ij}(N_{ij}^{paid})$$

- $Y_{ij}(k)$ i.i.d., independent of number of claims, independent of reporting and payment delay

Double chain-ladder

„Distribution-free“ formulation

$\Delta_m = \{X_{ij} : (i,j) \in \mathcal{I}\}$ **traingle of claims paid**

$\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}\}$ **traingle of incurred claims counts**

- Claim is not usually paid immediately after notification. This motivates the introduction of the third triangle.

N_{ijl}^{paid} **– number of payments from N_{ij} claims with l periods delay after being reported,** $l = 0, \ldots, m\text{-}1$**;**

N_{ij}^{paid} **– number of claims incurred in period i and (fully) paid with j periods delay**

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,0,j}^{paid}$$

Assumptions

- N_{ij} random variables with mean having a multiplicative parametrization $E[N_{ij}] = \alpha_i \beta_j$ and identification $\Sigma\, \beta_j = 1$ (Mack)
- The mean of the RBNS delay variables is $E[N_{ijl}^{paid} \mid \aleph_m] = N_{ij}\tilde{\pi}_l$ for each $(i,j) \in \mathcal{I}$ and $l = 0, \ldots, m\text{-}1$;
- Claim may be settled with several payments $Y_{ijl}(k)$. Conditional on the number of payments, the mean of individual payment size is given by $E[Y_{ijl}(k) \mid N_{ijl}^{paid}] = \tilde{\mu}_l \gamma_i$. (Note that $E[Y_{ijl}(k) \mid N_{ijl}^{paid}]$ does not depend on j – reporting delay.)

Main differences

- Assumptions are written in terms of first moments, rather than in terms of underlying distributional assumptions
- Model allows multiple payments per claim
  - Authors argued that it is rather difficult to specify a proper distribution in case that multiple payments are allowed – thus this feature has a limited use when one is interested in full distribution (bootstrapping) and not only best estimate.

Basic idea: DCL estimates derived through comparison of theoretical unconditioned means of claim counts and claim payments calculated from the underlying DCL assumptions and ordinary chain-ladder.

Using the DCL assumptions, we have

$$\mathrm{E}\left[\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)} \,|\, \aleph_m\right] = \mathrm{E}\left[\sum_{k=1}^{N_{i,j-l,l}^{paid}} \mathrm{E}[Y_{i,j-l,l}^{(k)} \,|\, \aleph_m, N_{i,j-l,l}^{paid}] \,|\, \aleph_m\right]$$

$$= \mathrm{E}[N_{i,j-l,l}^{paid}\, \tilde{\mu}_l \gamma_i \,|\, \aleph_m] = N_{i,j-l}\, \tilde{\pi}_l\, \tilde{\mu}_l \gamma_i$$

And since the aggregate payments can be written as

$$X_{ij} = \sum_{l=0}^{j} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}, \quad \text{for each} \ (i,j) \in \mathcal{I}$$

we have …

… for the conditional and unconditional means

$$\mathrm{E}[X_{ij} \mid \aleph_m] = \sum_{l=0}^{j} N_{i,j-l}\,\tilde{\pi}_l\,\tilde{\mu}_l\,\gamma_i = \sum_{l=0}^{j} N_{i,j-l}\,\pi_l\,\mu\gamma_i$$

$$\mathrm{E}[X_{ij}] = \alpha_i\,\mu\gamma_i \sum_{l=0}^{j} \beta_{j-l}\,\pi_l$$

where

$$\mu = \textstyle\sum_{l=0}^{m-1} \tilde{\pi}_l\,\tilde{\mu}_l$$

$$\pi_l = \tilde{\pi}_l\,\tilde{\mu}_l / \mu.$$

- **It is possible to use both conditional and unconditional mean to estimate „RBNS“ part**
- **It is possible to use unconditional mean to estimate „IBNR“ part**
- **Parameters α_i, β_j can be estimated using ordinary chain-ladder applied on the triangle of claim counts**
- **It remains to estimate μ, γ_i and π_l.**

Ordinary chain-ladder assumptions applied on the triangle of claims paid say there exist parameters $\widetilde{\alpha}_i, \widetilde{\beta}_j$, so that it is satisfied

$$\mathrm{E}[X_{ij}] = \widetilde{\alpha}_i \widetilde{\beta}_j$$

A direct comparison with the previous formula

$$\mathrm{E}[X_{ij}] = \alpha_i \mu \gamma_i \sum_{l=0}^{j} \beta_{j-l} \pi_l$$

leads to a natural identification

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i$$

$$\sum_{l=0}^{j} \beta_{j-l} \pi_l = \widetilde{\beta}_j$$

- **Parameters α_i, β_j and $\widetilde{\alpha}_i, \widetilde{\beta}_j$ can be estimated using the ordinary chain-ladder method on the triangles of incurred counts and claims paid. Let us denote these estimates by $\widehat{\alpha}_i, \widehat{\beta}_j$ and $\widehat{\widetilde{\alpha}}_i, \widehat{\widetilde{\beta}}_j$.**
- **They can be used for estimates of μ, γ_i and π_l in the following way.**

The second identification formula

$$\sum_{l=0}^{j} \beta_{j-l}\pi_l = \widetilde{\beta}_j$$

allows to estimate π_l using estimates of the other two coefficients, $\hat{\beta}_j$ and $\hat{\tilde{\beta}}_j$.

For the estimate of π_l , one then needs to solve a linear system

$$\begin{pmatrix} \widehat{\widetilde{\beta}}_0 \\ \vdots \\ \vdots \\ \widehat{\widetilde{\beta}}_{m-1} \end{pmatrix} = \begin{pmatrix} \hat{\beta}_0 & 0 & \cdots & 0 \\ \hat{\beta}_1 & \hat{\beta}_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \hat{\beta}_{m-1} & \cdots & \hat{\beta}_1 & \hat{\beta}_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}$$

Let us denote the solutions by $\hat{\pi}_l$.

Other parameters can be estimated using the first identification formula

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i$$

by estimates of the other two coefficients, $\hat{\alpha}_i$ and $\hat{\tilde{\alpha}}_i$, got from ordinary chain-ladder and using the formula

$$\widehat{\gamma}_i = \frac{\widehat{\widetilde{\alpha}}_i}{\widehat{\alpha}_i \widehat{\mu}}$$

since it is natural to put $\gamma_1 = 1$ and estimate

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}$$

✓ **Estimates of all parameters complete**

Double chain-ladder

Reminder

For the conditional and unconditional means, we have

$$\boxed{\mathrm{E}[X_{ij} \mid \aleph_m] = \sum_{l=0}^{j} N_{i,j-l}\, \tilde{\pi}_l\, \tilde{\mu}_l\, \gamma_i = \sum_{l=0}^{j} N_{i,j-l}\, \pi_l\, \mu\gamma_i}$$

$$\boxed{\mathrm{E}[X_{ij}] = \alpha_i\, \mu\gamma_i \sum_{l=0}^{j} \beta_{j-l}\, \pi_l}$$

where

$$\boxed{\mu = \sum_{l=0}^{m-1} \tilde{\pi}_l\, \tilde{\mu}_l}$$

$$\boxed{\pi_l = \tilde{\pi}_l\, \tilde{\mu}_l / \mu}$$

- **It is possible to use both conditional and unconditional mean to estimate the „RBNS“ part**
- **It is possible to use unconditional mean to estimate the „IBNR“ part**

Two possible estimates for „RBNS“ component and one for „IBNR“ component:

$$\widehat{X}_{ij}^{rbns(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i$$

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i$$

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\,\hat{\pi}_l\,\hat{\mu}\,\hat{\gamma}_i$$

where

$$\widehat{N}_{ij} = \hat{\alpha}_i\,\widehat{\beta}_j$$

Two possible estimates for „RBNS“ component and one for „IBNR“ component:

$$\widehat{X}_{ij}^{rbns(1)} = \sum_{l=i-m+j}^{j} \boxed{N_{i,j-l}}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i$$

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \boxed{\widehat{N}_{i,j-l}}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i$$

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\,\hat{\pi}_l \hat{\mu} \hat{\gamma}_i$$

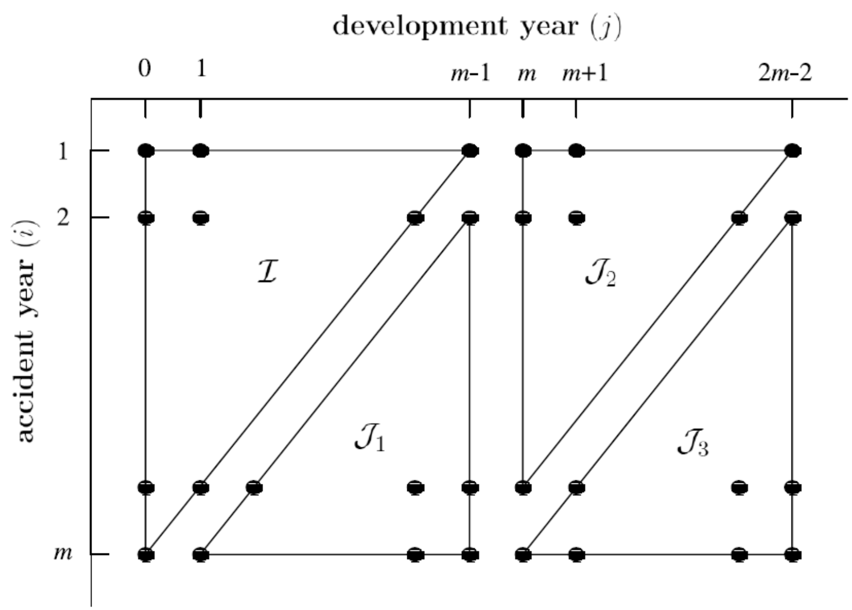
where

$$\widehat{N}_{ij} = \hat{\alpha}_i \hat{\beta}_j$$

It is also possible to include an estimate of tail

$$\widehat{R}^{tail} = \sum_{(i,j)\in\mathcal{J}_2\cup\mathcal{J}_3} \sum_{l=0}^{\min(j,d)} \widehat{N}_{i,j-l}\,\widehat{\pi}_l\,\widehat{\mu}\,\widehat{\gamma}_i$$

- The credibility of the estimate relies heavily on the fact whether „the full run-off" is observed in the first accident year



Two main features in which DCL differs from ordinary CL

- Estimate considers not only the lower triangle but also the tail
- Two possible estimates for the „RBNS“ part
  - The first one feels more natural as it uses *the true observed value*
  - Using the second one and ignoring the tail, we arrive exactly to the *ordinary chain-ladder estimate*
  - Both estimates will be close to each other if there is little difference between N_{ij} and $\widehat{N}_{ij}$

Distribution-free assumptions

- Underlying model does not rely on specific distributional assumptions

Main disadvantage

- **First moment formulation suitable only for the best estimate**
- **Proposed solution: fit a parametric model and use a parametric bootstrapping**

Triangle of counts

i\j	0	1	2	3	4	5	6	7	8	9
1	6238	831	49	7	1	1	2	1	2	3
2	7773	1381	23	4	1	3	1	1	3	
3	10306	1093	17	5	2	0	2	2		
4	9639	995	17	6	1	5	4			
5	9511	1386	39	4	6	5				
6	10023	1342	31	16	9					
7	9834	1424	59	24						
8	10899	1503	84							
9	11954	1704								
10	10989									

Triangle of paid claims (adjusted to calendar inflation)

i\j	0	1	2	3	4	5	6	7	8	9
1	451288	339519	333371	144988	93243	45511	25217	20406	31482	1729
2	448627	512882	168467	130674	56044	33397	56071	26522	14346	
3	693574	497737	202272	120753	125046	37154	27608	17864		
4	652043	546406	244474	200896	106802	106753	63688			
5	566082	503970	217838	145181	165519	91313				
6	606606	562543	227374	153551	132743					
7	536976	472525	154205	150564						
8	554833	590880	300964							
9	537238	701111								
10	684944									

Double chain-ladder

Case study – best estimate

	DCL			MNNV			
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	CL
1	1260	97	1357	1307	93	1399	1354
2	672	83	754	720	78	798	754
3	453	35	489	494	34	529	489
4	292	26	319	323	26	349	318
5	165	20	185	188	20	208	185
6	103	12	115	117	12	130	115
7	54	9	63	65	9	74	63
8	30	5	36	37	5	42	36
9	0	5	5	0	6	6	2
10	1		1		1	1	
11	0.6		0.6		0.6	0.6	
12	0.4		0.4		0.4	0.4	
13	0.2		0.2		0.2	0.2	
14	0.1		0.1		0.1	0.1	
15	0.06		0.06		0.07	0.07	
16	0.03		0.03		0.04	0.04	
17	0.01		0.01		0.02	0.02	
Total	3030	296	3326	3251	287	3538	3316

Case study
First part

Best estimate in “distribution-free” DCL

Case study: first part

Best estimate in “distribution-free” DCL

Data

- 33 triangles
- Based on data observed in different lines of business (MTPL, TPL, Casco, Property, Travel, Accident, Sickness, Property)
  - 6 long tailed
  - 27 short tailed
  - No salvages & subrogations

Analysis based on the ordinary chain-ladder as a benchmark

- Concerning the best estimate, there are, in fact, only two sources of difference:
  - Using observed rather than “averaged” number of claims in the RBNS part
  - Estimate of tail

We did not apply any smoothing of development factors in the underlying chain-ladder estimates

- No strictly standardized methodology ➔ leading to arbitrary choices and possible misinterpretations

Results

- In majority of examined triangles (30 of 33), little difference between ordinary CL and DCL predictions
- Basic descriptive indicators (diff of total reserves by DCL and CL / total reserve by CL applied on paid triangle)

DCL without tail	Diff in %
Min	-5,7 %
Max	+2,1 %
Average	-0,6 %
StDev	+1,3 %

DCL with tail	Diff in %
Min	-5,5 %
Max	+12,1 %
Average	+0,5 %
StDev	+3,3 %

Results

- Tail relevant in long tailed lines of business (MTPL, TPL)
- For short lines of business, the influence of tail usually negligible

DCL without tail, short tailed LoBs	Diff in %
Min	-5,7 %
Max	+2,1 %
Average	-0,7 %
StDev	+1,4 %

DCL with tail, short tailed LoBs	Diff in %
Min	-5,5 %
Max	+2,1 %
Average	-0,7 %
StDev	+1,4 %

Case study: first part

Best estimate in "distribution-free" DCL (cont'd)

Results

- Tail relevant in long tailed lines of business (MTPL, TPL)
- For short lines of business, the influence of tail usually negligble

DCL without tail, long tailed LoBs	Diff in %
Min	-1,1 %
Max	+0,1 %
Average	-0,5 %
StDev	+0,4 %

DCL with tail, long tailed LoBs	Diff in %
Min	+0,2 %
Max	+12,1 %
Average	+5,2 %
StDev	+4,5 %

RBNS estimate

- Comparison of DCL estimate / case-by-case estimate
  - Total
  - Last accident period

Without several exceptions (Casco), it differs a lot in both cases – by tens of percent. Possible reasons:

- Run-off (case-by-case reserves are not BE)
  - In majority of cases, the DCL shows an indication of over/underreserving consistently with the run-off test
- RBNS in older periods: wrong tail, low number of still opened claims
- Real world ≠ best estimate
- Assumptions of DCL:
  - Average claim can differ in accident years but not in reporting periods (usually not satisfied)
  - One payment pattern cannot capture differences in accident years

Backtesting

- Test proposed by authors of the method
- Cut-off last 1-4 diagonals and compare the accuracy of CL / DCL predictions
- Potential weakness: better fit may be simply a coincidence

Results

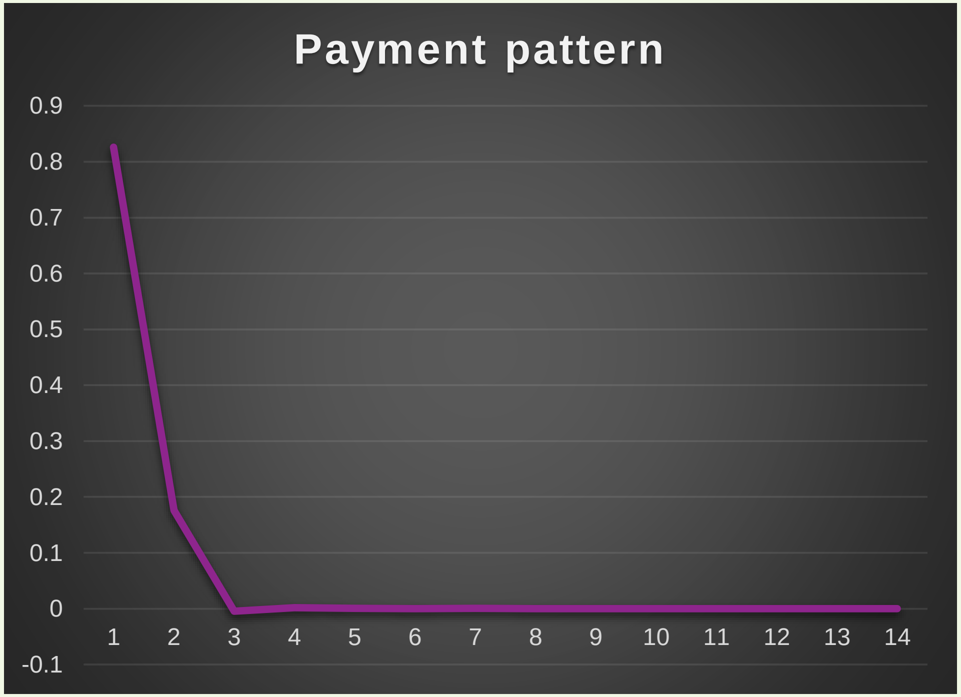
- DCL very slightly more precise (1.0 % on average, quite likely statistically insignificant)
- Test on 4 cut-off diagonals:
  - 19 of 30 triangles: difference less than 1 % on total reserves
  - 25 of 30 triangles: difference less than 3 % on total reserves
  - 27 of 30 triangles: difference less than 5 % on total reserves
  - 28 of 30 triangles: difference less than 10 % on total reserves
  - 2 remaining triangles: difference of 10.4 % and 22.9 % (DCL being more precise in both cases)

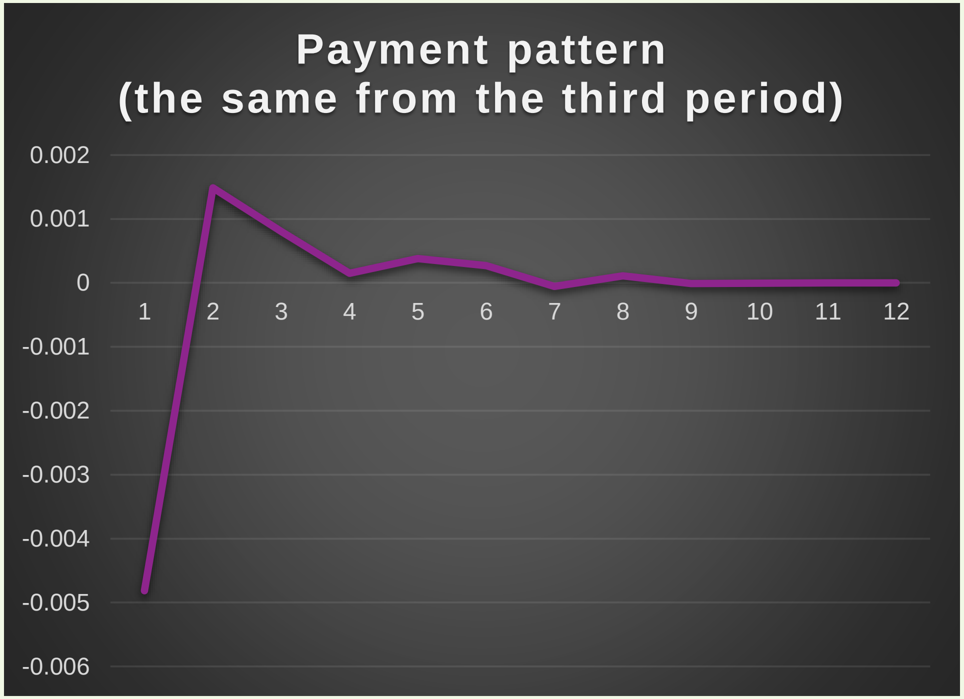
→ The accuracy of ordinary CL and DCL in the „naive“ approach is very similar

→ If one method is (in)accurate, so is the other

Charts

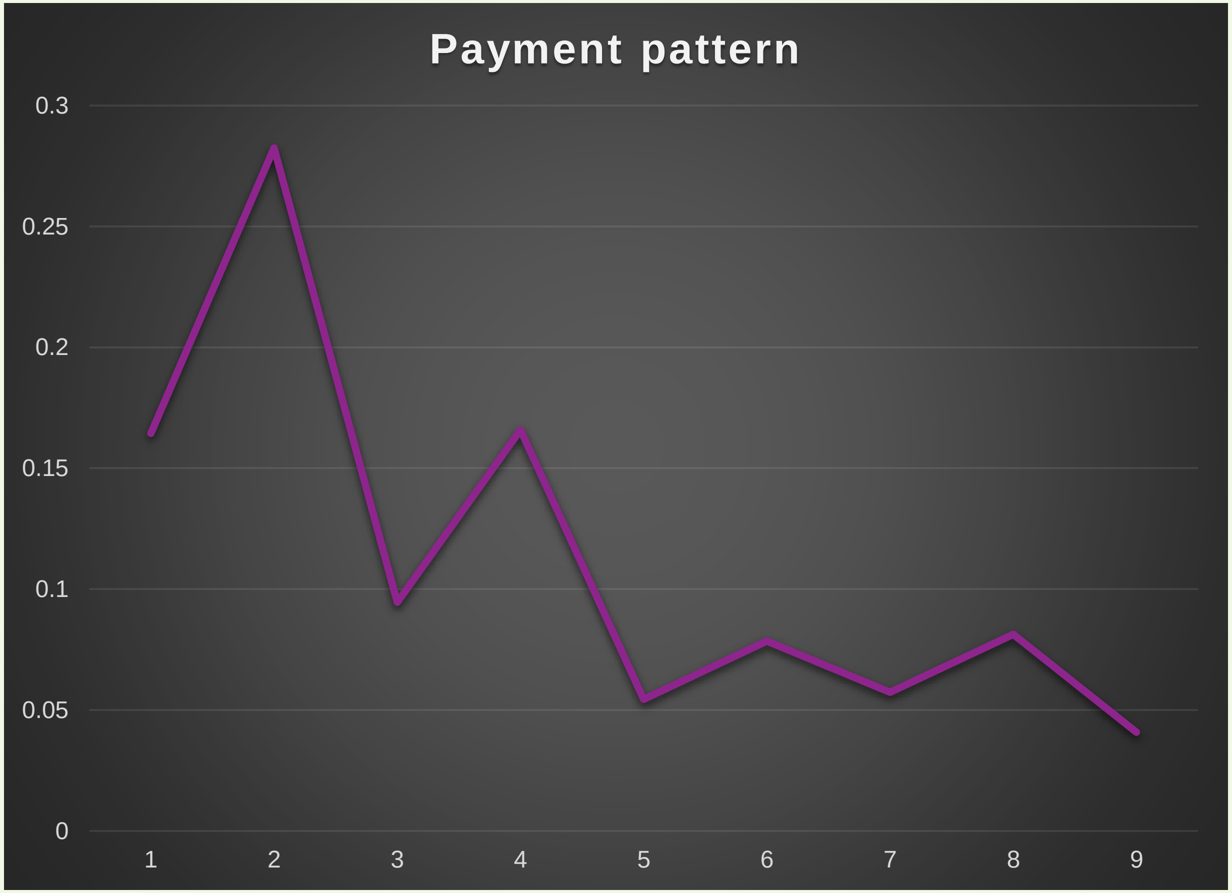
- Payment delay (factors π_j)
- Example of stable, short-tailed triangle





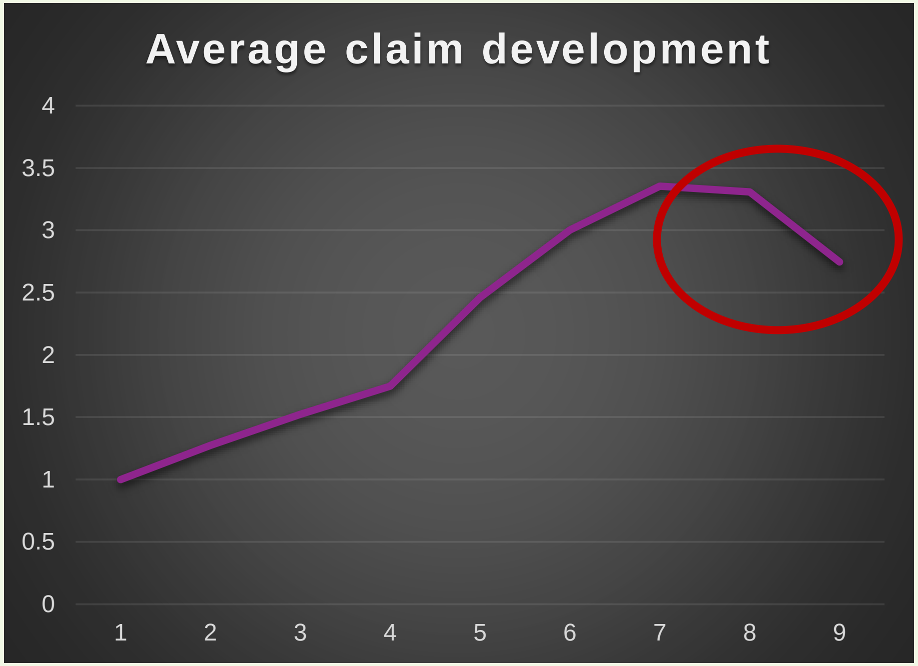
Charts

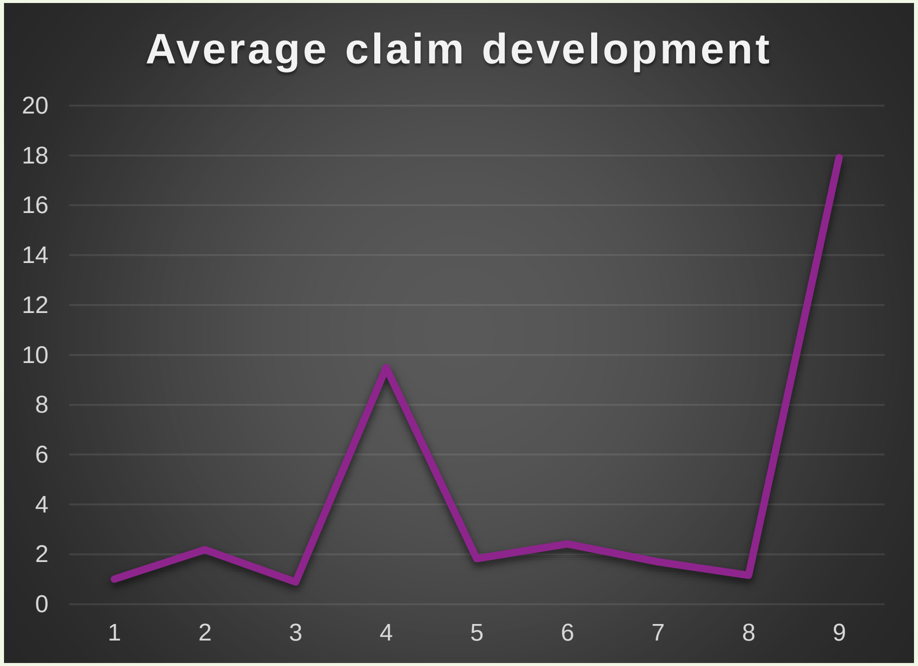
- Payment delay (factors π_j)
- Example of unstable and long-tailed triangle



Charts

- Inflation (factors γ_i)
- Chart on the left side: well-behaved development, irregularity in the most recent years
- Chart on the right side: either extreme volatility in average claim or the model does not fit well





Case study: first part
Diagnostics (cont'd)

Other

- Average claim amounts
  - Check on market data
- Number of claims
  - Does it correspond to the market share?
- Checks common in ordinary chain-ladder
  - Predicted claim ratios in accident years
  - Claims development pattern
  - Outliers
  - …

Except the reporting delay and payment delay patterns, other values can be compared to the market

Reporting delay and the payment delay can be inspected in detail through detailed data

Example of the parametric model

Parametric bootstrap

$\Delta_m = \{X_{ij} : (i,j) \in I\}$ **traingle of claims paid**

$\aleph_m = \{N_{ij} : (i,j) \in I\}$ **traingle of incurred claims counts**

- Claim is not usually paid immediately after notification. This motivates the introduction of the third triangle.

N_{ijk}^{paid} **– part of the N_{ij} claims fully paid with k periods delay after being reported,** $k = 0, \ldots, d, d \leq m\text{-}1$

N_{ij}^{paid} **– number of claims incurred in period i and (fully) paid with j periods delay**

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

Assumptions

- N_{ij} independent, with Poisson distribution (ML estimate leads to classical CL algorithm)
- Given N_{ij}, the numbers of payments follow a multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

- Claim settled with one payment. Thus, if we denote $Y_{ij}(k)$ the payment for the k-th claim incurred in period i settled with j periods delay, we have

$$X_{ij} = Y_{ij}(1) + Y_{ij}(2) + \ldots + Y_{ij}(N_{ij}^{paid})$$

- $Y_{ij}(k)$ are mutually independent, with distributions f_i, mean $\mu_i = \mu\gamma_i$ and variance $\sigma_i^2 = \sigma^2\gamma_i^2$
- $Y_{ij}(k)$ independent of number of claims, independent of reporting and payment delay

Individual claims distribution (severity distribution) may be chosen

- One possible choice is gamma distribution with the mean μ_i and the variance σ_i^2
- Thus it has the shape parameter $\lambda_i = \mu_i^2 / \sigma_i^2$ and the scale parameter $\kappa_i = \sigma_i^2 / \mu_i$.
- Given the count N_{ij}^{paid}, the aggregate claims X_{ij} are again gamma distributed with shape $N_{ij}^{paid}\lambda_i$ and scale κ_i (sum of identically gamma distributed random variables is again gamma distributed random variable)

➔ Need to estimate σ_i^2

➔ Need to estimate p_l, since the original estimate of parameters π_l may lead both to negative values and values which does not sum up to 1

From the estimate of π_l , one can estimate p_l by several ways, authors suggested two very simple methods

- Maximal delay d is estimated by summing the number of succesive estimates of π_l until a number greater or equal to one is achieved. Then d is equal to the count of summands and it is put

$$\begin{aligned}\widehat{p}_l &= \widehat{\pi}_l, l = 0, \ldots, d-1, \\ \widehat{p}_d &= 1 - \sum_{l=0}^{d-1} \widehat{p}_l.\end{aligned}$$

- Nullify negative π_l coefficients and then rescale them so that their sum would be equal to 1.

- In practice:
  - There should be (!) either way little difference between π_l and p_l;
  - It is advised to use the option which would less modify the best estimate.

Adjustments of the payment pattern can change the predictions quite a bit

- These were the results with general coefficients π_l

DCL without tail	Diff in %
Min	-5,7 %
Max	+2,1 %
Average	-0,6 %
StDev	+1,3 %

DCL with tail	Diff in %
Min	-5,5 %
Max	+12,1 %
Average	+0,5 %
StDev	+3,3 %

Double chain-ladder
Influence on the results of case studies (cont'd)

Adjustments of the payment pattern can change the predictions quite a bit

- This is how they change for the first adjustment ("cut-off")

DCL without tail	Diff in %
Min	-5,7 % → -24,8 %
Max	+2,1 % → +22,8 %
Average	-0,6 % → -2,0 %
StDev	+1,3 % → +7,3 %

DCL with tail	Diff in %
Min	-5,5 % → -23,4 %
Max	+12,1 % → +22,8 %
Average	+0,5 % → -1,2 %
StDev	+3,3 % → +7,1 %

Adjustments of the payment pattern can change the predictions quite a bit

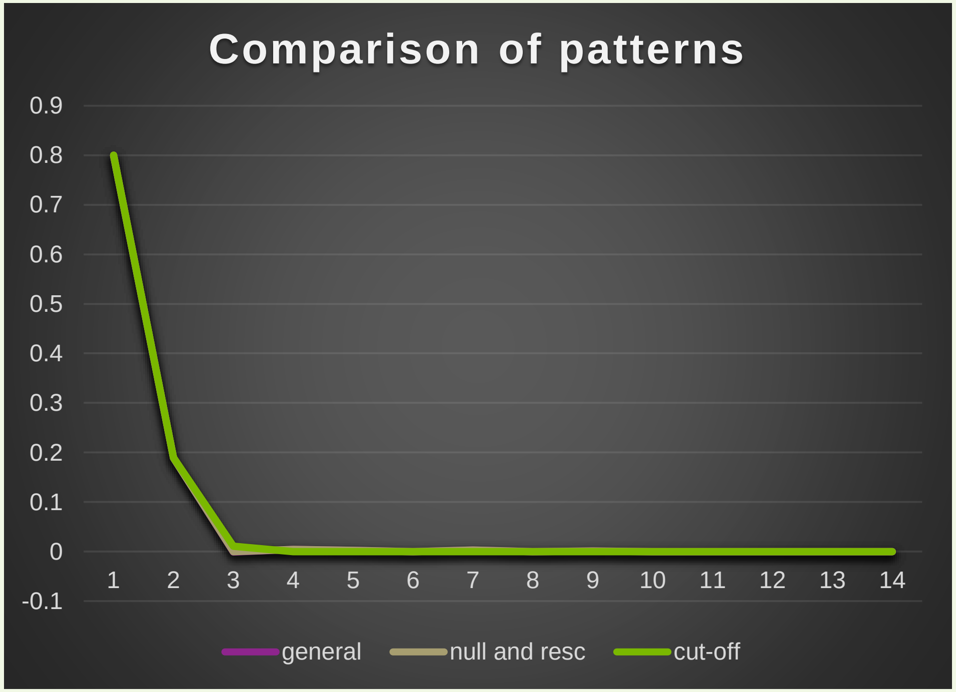
- This is how they change for the second adjustment ("nullifying and rescaling")

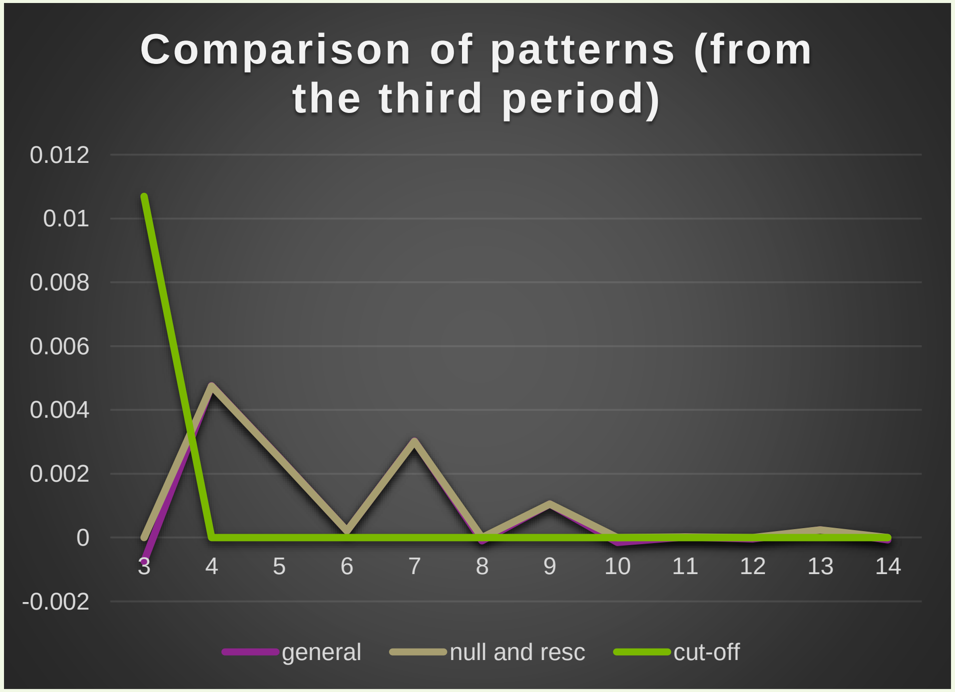
DCL without tail	Diff in %
Min	-5,7 % → -1,1 %
Max	+2,1 % → +51,5 %
Average	-0,6 % → +6,2 %
StDev	+1,3 % → +10,4 %

DCL with tail	Diff in %
Min	-5,5 % → -1,1 %
Max	+12,1 % → +51,5 %
Average	+0,5 % → +7,4 %
StDev	+3,3 % → +10,5 %

Differences between adjustments

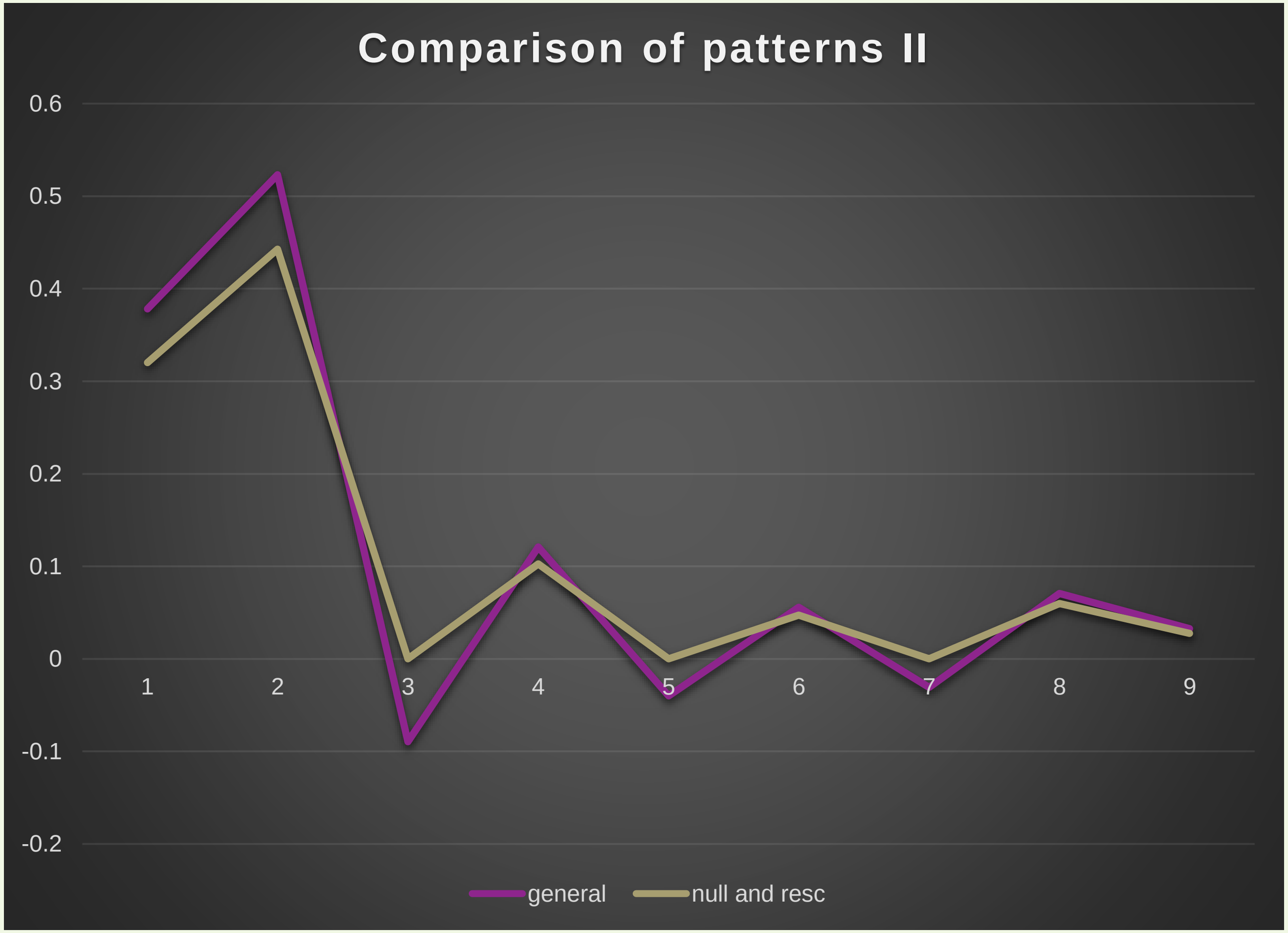
- Can be both small / large
- For this case, the difference between best estimates using "cut-off" and "nullify and rescale" adjustments was over 10 %
- *All three patterns coincide in the first two periods. Green and blue line almost coincides also for the following periods.*





Differences between adjustments

- In the previous slide, the adjustment by "nullifying and rescaling" was minor but this might not be the case every time



For bootstrap, we also need the estimates of the variance parameters.

Estimate of variance parameters is based on the fact, that the claims amounts follow (approximately) over-dispersed Poisson model

- See Appendix A for details

$$\mathrm{V}[X_{ij} | \aleph_m] \approx \frac{\sigma_i^2 + \mu_i^2}{\mu_i} \mathrm{E}[X_{ij} | \aleph_m]$$

$$= \gamma_i \frac{\sigma^2 + \mu^2}{\mu} \mathrm{E}[X_{ij} | \aleph_m]$$

$$= \varphi_i \mathrm{E}[X_{ij} | \aleph_m]$$

where

$$\varphi_i = \gamma_i \varphi$$

$$\varphi = \frac{\sigma^2 + \mu^2}{\mu}$$

We can estimate the over-dispersion parameter using the over-dispersion (Pearson X^2) statistics

$$\widehat{\varphi} = \frac{1}{n-(d+1)} \sum_{i,j \in \mathcal{I}} \frac{(X_{ij} - \widehat{X}_{ij}^{DCL})^2}{\widehat{X}_{ij}^{DCL} \widehat{\gamma}_i}$$

Where

$$n = m(m+1)/2$$

and

$$\widehat{X}_{ij}^{DCL} = \sum_{l=0}^{\min(j,d)} N_{i,j-l}\, \hat{p}_l \hat{\mu} \hat{\gamma}_i$$

The variance factors are then estimated by

$$\widehat{\sigma}^2 = \widehat{\mu}\widehat{\varphi} - \widehat{\mu}^2$$

$$\widehat{\sigma}_i^2 = \widehat{\sigma}^2 \widehat{\gamma}_i^2$$

RBNS part of the reserve

Reported counts: left-top triangle I

- The actual values N_I are observed

Aggregated claims X_{ij} arising from (already) reported claims (triangles I u J_1 u J_2, only I is observed)

- Constructed sequentially:
  - Given reported counts N_{ij}, number of payments N_{ijk}^{paid} follows the multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

  - The paid counts N_{ij}^{paid} are defined by

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,j-\min(j,d),\min(j,d)}^{paid}$$

  - Individual claims distribution (severity distribution) chosen as gamma distribution

IBNR part of the reserve

Incurred but not yet reported counts: right-bottom triangle J_1

- Poisson distribution N_{J1}

Aggregated claims X_{ij} arising from incurred but not yet reported claims (triangles J_1 u J_2 u J_3)

- Constructed analogically to the previous "RBNS case", given the prediction of claims counts N_{J1}

Double chain-ladder

Bootstrapping – process variance + parameter estimation error

Process variance (stochastic error) only

- Simulation of unknown parts of the triangles (bottom-right + tail) from estimated parameters
  – RBNS part: simulate claims payments using the previous construction
  – IBNR part: simulate number of claims in the triangle J_1 and, based on this, simulate claims payments as in the RBNS part
  – In these simulations, parameters of the underlying distributions are fixed (except the simulated claims counts in the IBNR part)

Process variance and parameter estimation errors

- Estimated parameters used for a simulation of new „left-top“ triangle(s)
  – RBNS part: only paid triangle (as I use observed values in triangle of counts for the estimate of the RBNS)
  – IBNR part: both triangle of payments and triangle of counts
- From these new triangles, “bootstrapped“ parameters are estimated
- From “bootstrapped“ parameters, the unknown parts of triangles are simulated

Proposed algorithm for the bootstrapping procedure – RBNS part

Estimate of process variance only – do only steps 1, 4 and 5 (using parameters estimated in the step 1).

1. Parameters estimation

- Apply the procedure described for the best estimate to obtain estimates for p, μ, σ^2, λ, κ

2. Bootstrapping the data

- Keep the same counts N, but bootstrap the aggregate payments X* as follows
  - Simulate the delay (construct N_{ij}^{paid*} from given N_{ij} using the multinomial distribution estimated in the step 1)
  - Simulate the aggregate payments using gamma distribution with shape parameter $N_{ij}^{paid*}\lambda$ and scale parameter κ

3. Bootstrapping the parameters

- From the bootstrap data (N, X*) generated at step 2 obtain new estimates for p^*, μ^*, σ^{2*}, λ^*, κ^*

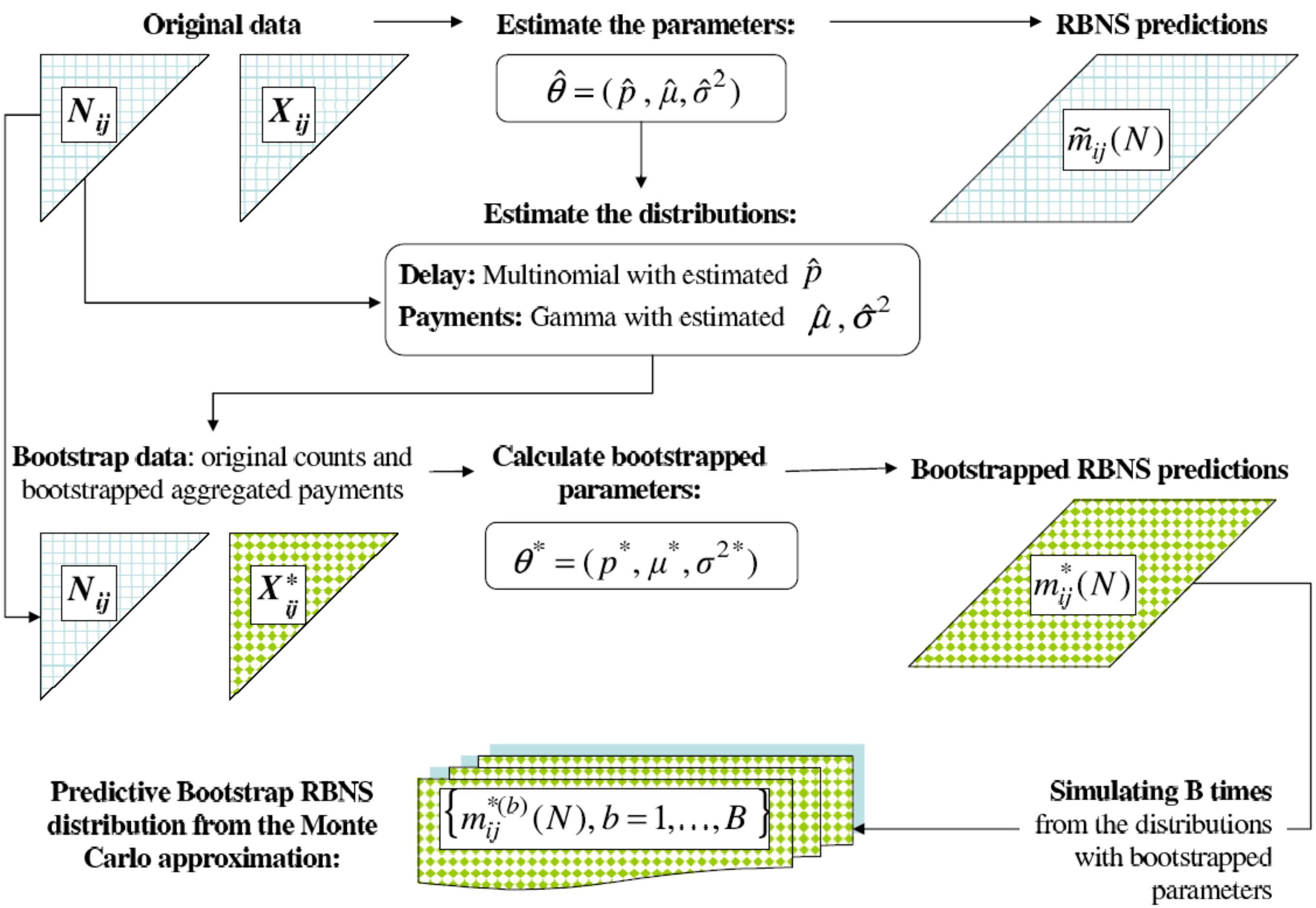
4. Bootstrapping the RBNS prediction

- Simulate the delay as in the step 2
- Simulate the aggregate payments as in the step 2
- Get the bootstrapped RBNS prediction

5. Monte Carlo approximation

- Repeat steps 2-4 B times and get the empirical bootstrap distribution of the RBNS part of the reserve

Algorithm RBNS – Bootstrapping taking into account the uncertainty parameters



Proposed algorithm for the bootstrapping procedure – IBNR part

1. Parameters and distribution estimation

- Apply the procedure described for the best estimate to obtain estimates for p, μ, σ^2, λ, κ and use the chain-ladder to estimate future incurred claims counts (ω).

2. Bootstrapping the data

- Get new counts N^* and aggregate payments X^* as follows
  - Simulate new counts N^* (in the upper-left triangle) using Poisson distribution (with parameters estimated by the chain-ladder method in the step 1)
  - Using N^*, simulate X^* as in the second step of the RBNS procedure

3. Bootstrapping the parameters

- From the bootstrap data (N^*, X^*) generated at step 2 obtain new estimates for p^*, μ^*, σ^{2*}, λ^*, κ^* and use the chain-ladder to get bootstrapped future incurred claims counts.

4. Bootstrapping the RBNS prediction

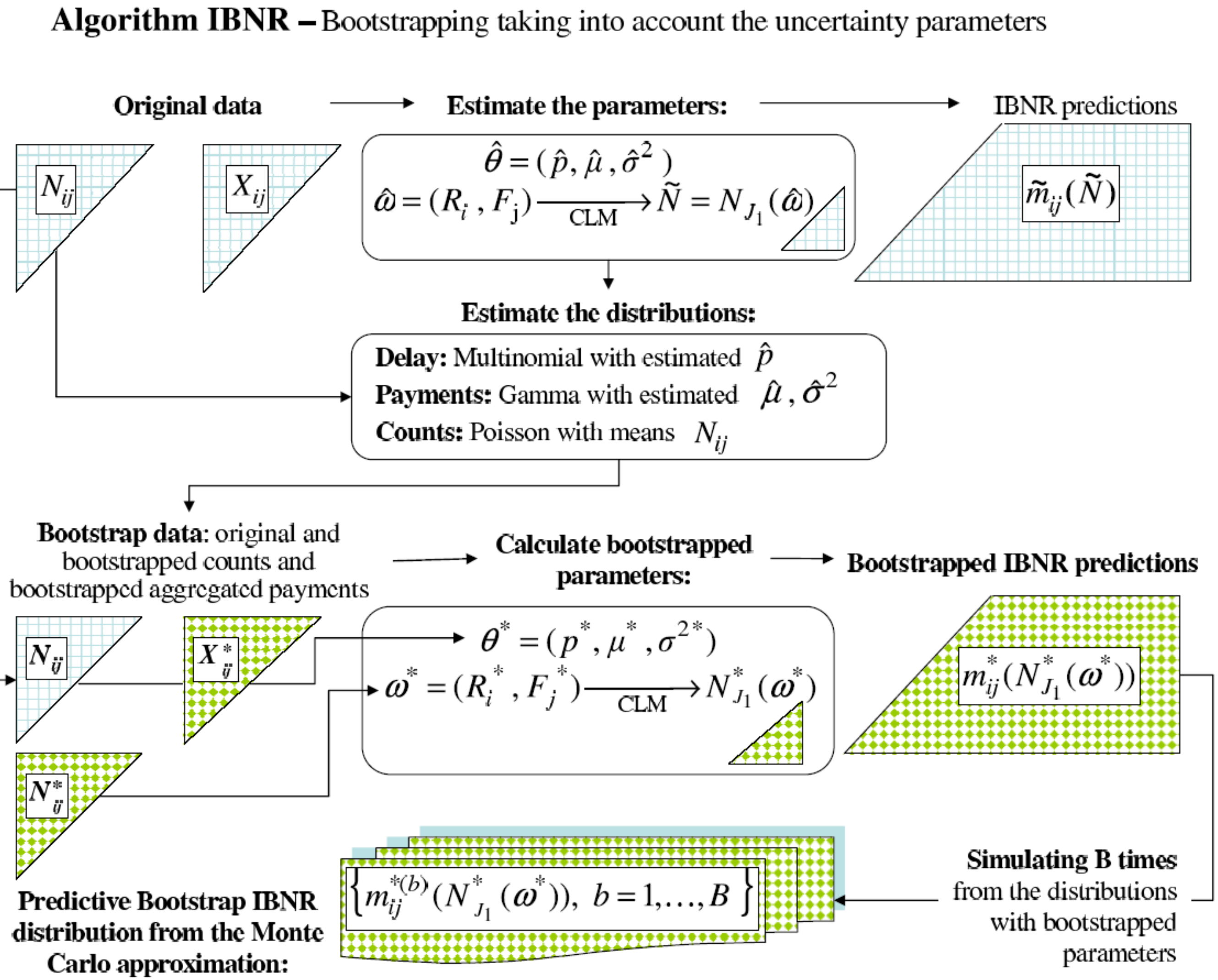
- Simulate the delay for N_{ij}^* using p^*, i.e. construct $N_{ij}^{paid*,\ IBNR}$ analogously to the step 2 of the “RBNS” procedure
- Simulate the aggregate payments as in the step 2 and get the bootstrapped IBNR prediction (an. “RBNS” procedure)

5. Monte Carlo approximation

- Repeat steps 2-4 B times and get the empirical bootstrap distribution of the IBNR part of the reserve

Double chain-ladder

Bootstrapping – algorithm schema (IBNR part)



Double chain-ladder

Case study provided in the paper (MTPL)

Triangle of counts

i\j	0	1	2	3	4	5	6	7	8	9
1	6238	831	49	7	1	1	2	1	2	3
2	7773	1381	23	4	1	3	1	1	3	
3	10306	1093	17	5	2	0	2	2		
4	9639	995	17	6	1	5	4			
5	9511	1386	39	4	6	5				
6	10023	1342	31	16	9					
7	9834	1424	59	24						
8	10899	1503	84							
9	11954	1704								
10	10989									

Triangle of paid claims (adjusted to calendar inflation)

i\j	0	1	2	3	4	5	6	7	8	9
1	451288	339519	333371	144988	93243	45511	25217	20406	31482	1729
2	448627	512882	168467	130674	56044	33397	56071	26522	14346	
3	693574	497737	202272	120753	125046	37154	27608	17864		
4	652043	546406	244474	200896	106802	106753	63688			
5	566082	503970	217838	145181	165519	91313				
6	606606	562543	227374	153551	132743					
7	536976	472525	154205	150564						
8	554833	590880	300964							
9	537238	701111								
10	684944									

MNNV

- Predecessor of the double chain-ladder
- Does not consider inflation in the accident year direction
- Uses different estimate procedure – maximization of quasi log-likelihood function

	DCL			MNNV			
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	CL
1	1260	97	1357	1307	93	1399	1354
2	672	83	754	720	78	798	754
3	453	35	489	494	34	529	489
4	292	26	319	323	26	349	318
5	165	20	185	188	20	208	185
6	103	12	115	117	12	130	115
7	54	9	63	65	9	74	63
8	30	5	36	37	5	42	36
9	0	5	5	0	6	6	2
10	1		1		1	1	
11	0.6		0.6		0.6	0.6	
12	0.4		0.4		0.4	0.4	
13	0.2		0.2		0.2	0.2	
14	0.1		0.1		0.1	0.1	
15	0.06		0.06		0.07	0.07	
16	0.03		0.03		0.04	0.04	
17	0.01		0.01		0.02	0.02	
Total	3030	296	3326	3251	287	3538	3316

Double chain-ladder

Case study – bootstrap

	Bootstrap predictive distribution						
	DCL			MNNV			
	RBNS	IBNR	Total	RBNS	IBNR	Total	CL
mean	3013	294	3307	3134	274	3408	3314
pe	279	52	300	327	60	340	345
1%	2415	198	2661	2464	148	2714	2588
5%	2575	215	2821	2646	183	2895	2780
50%	2995	289	3291	3105	272	3390	3287
95%	3505	389	3813	3722	378	4002	3911
99%	3649	425	4020	3987	435	4275	4061

Case study
Second part

DCL bootstrapping

Data

- Same as in the first part. Tested on 30 triangles with small differences between CL and DCL best estimates.

Comparison with

- Mack's estimate of mean squared error of prediction (MSEP)
- Chain-ladder "two-stage" bootstrap method by Verall and England (1999) and England (2001)
  - Resampling residuals
  - Simulating payments from ODP distribution

Case study: second part

DCL bootstrapping (cont'd)

Results: coefficient of variation (cumulative figures)

| MSEP^(1/2) / BE |	< 5%	< 10%	< 15%	< 20%	< 25%
CL bootstrap	1	9	15	21	23
Mack analytic estimate	0	6	11	17	20
DCL, "cut-off"	0	3	9	12	17
DCL, "nullifying and rescaling"	0	12	17	21	24

| MSEP^(1/2) / BE |	< 25%	< 30%	< 40%	< 50%	≥ 50%
CL bootstrap	23	26	27	28	2
Mack analytic estimate	20	25	26	26	4
DCL, "cut-off"	17	18	19	20	10
DCL, "nullifying and rescaling"	24	24	24	25	5

Results: coefficient of variation, differences to CL bootstrap (incremental figures)

- **Number of triangles where the difference (in % of best estimate) between CL bootstrap (by England) and considered alternatives falls in the range:**

MSEP^(1/2), differences to CL bootstrap	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
Mack analytic estimate	16	9	3	1	1
DCL, "cut-off"	11	3	5	3	8
DCL, "nullifying and rescaling"	21	3	1	1	4

Case study: second part
DCL bootstrapping (cont'd)

Results: VaR 95% and VaR 99% (incremental figures)

- Unlike for best estimates, the results of bootstrap can differ quite significantly
- **Number of triangles where the difference (in % of best estimate) between CL and DCL bootstraps falls in the range:**

VaR 95%, differences to CL bootstrap	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, "cut-off"	12	2	6	0	10
DCL, "nullifying and rescaling"	16	6	2	1	5

VaR 99%, differences to CL bootstrap	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, "cut-off"	10	3	0	5	12
DCL, "nullifying and rescaling"	13	7	4	0	6

- Adjustments in DCL payment delay parameters ("cut-off" or "nullifying and rescaling") led in several cases to unrealistic estimates, but such an estimate was rarely seen for both adjusting methods at once

Case study: second part
DCL bootstrapping (cont’d)

Results: VaR 95% and VaR 99% (incremental figures)

- Adjustments in DCL payment delay parameters (“cut-off” or “nullifying and rescaling”) led in several cases to unrealistic estimates, but such an estimate was rarely seen for both adjusting methods at once
  - If we take the estimate “closer” to the chain-ladder bootstrap

VaR 95%, diff.	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, “cut-off”	12	2	6	0	10
DCL, “nullifying and rescaling”	16	6	2	1	5
DCL, choosing closer est.	17	7	2	0	4

VaR 99%	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, “cut-off”	10	3	0	5	12
DCL, “nullifying and rescaling”	13	7	4	0	6
DCL, choosing closer est.	16	6	3	1	4

External data in DCL method

Accident year inflation parameter

DCL and Bornhuetter-Ferguson

M.D.Martínez-Miranda, J. P. Nielsen, R. Verrall

April 2011 (preliminary), NAAJ 2013

Main difference

- Estimate of the inflation parameter using triangle of incurred claims

The name “Bornhuetter-Ferguson” is chosen simply because a (distant) resemblance

- Classical BF method replace the chain-ladder estimate of ultimate claim by a prior estimate derived differently

$$\widehat{\widehat{C_{i,J}}}^{\mathrm{BF}} = C_{i,I-i} + \left(1 - \widehat{\beta}_{I-i}^{(\mathrm{CL})}\right) \widehat{\mu}_i$$

$$\widehat{C_{i,J}}^{\mathrm{CL}} = C_{i,I-i} + \left(1 - \widehat{\beta}_{I-i}^{(\mathrm{CL})}\right) \widehat{C_{i,J}}^{\mathrm{CL}}$$

- The proposed adjustment to the DCL method is similar in the sense, that an inflation parameter derived by the DCL algorithm replaces by the inflation parameter derived differently *(but not completely deliberately, in fact, the DCL algorithm is simply not applied on the paid triangle but on the incurred triangle instead)*

Inflation parameter is estimated using the identification formula

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i$$

where alpha coefficients are estimated using the chain-ladder method on triangles of incurred counts (without tilde “~”) and claims paid (with tilde “~”). Then we can estimate

$$\widehat{\gamma}_i = \frac{\widehat{\widetilde{\alpha}}_i}{\widehat{\alpha}_i \widehat{\mu}}$$

and since the model is over-parametrized, this is solved by putting $\gamma_1 = 1$ and

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}$$

Two step procedure:

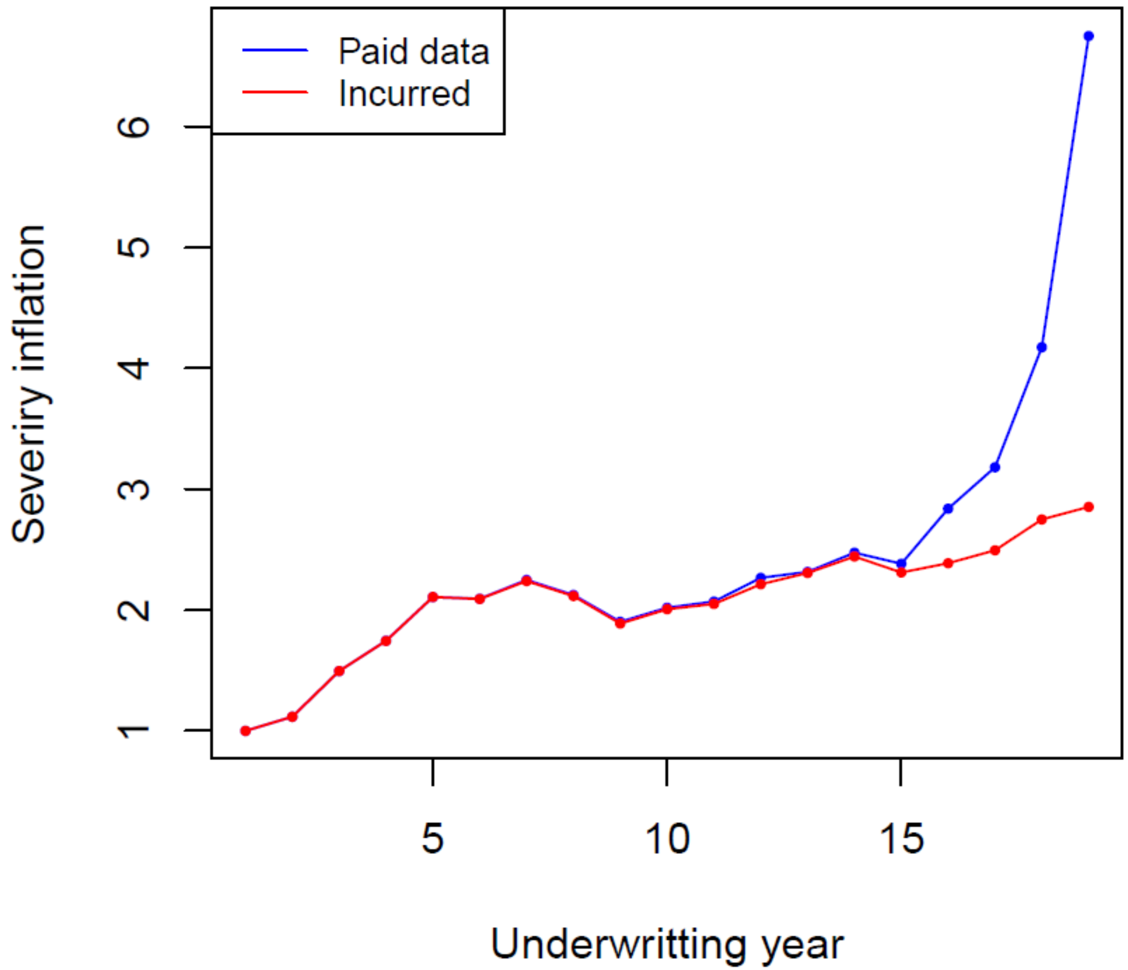
Parameter estimation

- Estimate all parameters (π, $\mathbf{p}$, μ, γ, σ) using the DCL procedure.
- Note that γ parameters estimated here using the paid triangles are implicitly used for the estimate of σ!

Proposed adjustment

- Repeat the estimation using the incurred triangle instead of the paid one
- Replace only the inflation parameters γ

Personal accident data "from major insurer" – 19 accident years



BDCL

Case study (cont'd)

	BDCL			DCL			
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	CCL
1	37812	615	38427	59844	1387	61230	61091
2	25878	3294	29171	41446	7406	48852	48061
3	17804	2537	20340	31015	5611	36626	36266
4	9485	2495	11980	17542	5501	23043	22990
5	3699	1867	5566	6443	4069	10512	10439
6	1839	821	2660	3192	1720	4912	4914
7	905	462	1366	1446	945	2390	2380
8	512	246	758	675	487	1162	1174
9	457	113	571	642	210	853	848
10	329	87	416	424	169	592	600
11	337	40	377	536	72	608	594
12	242	49	292	404	99	504	496
13	163	37	200	335	74	409	397
14	28	46	73	60	97	157	136
15	0	18	18	0	37	37	109
16	0	7	7	0	12	12	0
17	0	4	4	0	7	7	0
18	0	2	2	0	4	4	0
19	0	1	1	0	2	2	
20	0	1	1	0	1	1	
21	0	0	0	0	1	1	
22	0	0	0	0	0	0	
Total	99490	12741	112231	164003	27910	191913	190496

Best estimates differ dramatically

- Which result is more reliable?

Back-testing

- Compare predictions based on triangles with deleted last 1, 2, 3, … diagonals with reality
- Authors did the back-testing on quarterly triangles

Results of the Back-Test to Evaluate of the Discrepancy between Estimates and Actual Numbers

m_c	DCL	BDCL	Rerr
78	221,665.5	99,071.9	0.4469
77	210,708.1	98,297.5	0.4665
76	233,875.4	84,232.2	0.3602
75	317,434.6	77,075.6	0.2428
74	283,276.9	87,542.4	0.3090

Note: The second and third columns show the (square root) mean squared error of the estimates by DCL and BDCL, respectively. The discrepancies have been evaluated on the last $m - m_c$ diagonals in the original quarterly paid triangle. The last column shows the relative error defined as the ratio of the BDCL and the DCL errors.

Case study

Third part

BDCL: best estimate and bootstrapping

Data

- Same as in the first part.

Similar statistics as in the first two parts for DCL

Best estimate: BDCL, “moment” type best estimates

- **Table below summarizes both DCL and BDCL compared to ordinary CL (applied on paid triangle)**
- There are significant differences ➔ they are related to the differences between the prediction of claims reserves from paid and incurred triangles

Without tail	DCL to CL	BDCL to CL
Min	-5,7 %	-54,1 %
Max	+2,1 %	+20,2 %
Average	-0,6 %	-4,1 %
StDev	+1,3 %	+13,3 %

With tail	DCL to CL	BDCL to CL
Min	-5,5 %	-53,9 %
Max	+12,1 %	+23,0 %
Average	+0,5 %	-3,0 %
StDev	+3,3 %	+13,0 %

Case study: third part

Rel. differences of other estimates to the ordinary chain-ladder applied on paid triangles for 30 tested triangles (moment type variants of DCL and BDCL)

Adjustments of the payment pattern

- First type (“cut-off”)

DCL without tail	Diff in %
Min	-54,1 % → -58,1 %
Max	+20,2 % → +24,9 %
Average	-4,1 % → -5,4 %
StDev	+13,3 % → +16,1 %

DCL with tail	Diff in %
Min	-53,9 % → -58,1 %
Max	+23,0 % → +24,9 %
Average	-3,0 % → -4,7 %
StDev	+13,0 % → +15,8 %

Adjustments of the payment pattern

- Second type ("nullifying and rescaling")

DCL without tail	Diff in %
Min	-54,1 % → -38,0 %
Max	+20,2 % → +53,9 %
Average	-4,1 % → +2,8 %
StDev	+13,3 % → +17,1 %

DCL with tail	Diff in %
Min	-53,9 % → -37,8 %
Max	+23,0 % → +53,9 %
Average	-3,0 % → +3,9 %
StDev	+13,0 % → +16,6 %

Case study: second part
DCL bootstrapping (cont'd)

Results: coefficient of variation (cumulative figures)

| MSEP^(1/2) / BE |	< 5%	< 10%	< 15%	< 20%	< 25%
CL bootstrap	1	9	15	21	23
Mack analytic estimate	0	6	11	17	20
DCL, "cut-off"	0	3	9	12	17
DCL, "nullifying and rescaling"	0	12	17	21	24
BDCL, "cut-off"	0	3	7	10	15
BDCL, "nullifying and rescaling"	0	11	16	19	23

| MSEP^(1/2) / BE |	< 25%	< 30%	< 40%	< 50%	≥ 50%
CL bootstrap	23	26	27	28	2
Mack analytic estimate	20	25	26	26	4
DCL, "cut-off"	17	18	19	20	10
DCL, "nullifying and rescaling"	24	24	24	25	5
BDCL, "cut-off"	15	17	19	20	10
BDCL, "nullifying and rescaling"	23	24	24	25	5

Case study: second part

DCL bootstrapping (cont'd)

Results: coefficient of variation (cumulative figures)

| MSEP^(1/2) / BE |	< 5%	< 10%	< 15%	< 20%	< 25%
CL bootstrap	1	9	15	21	23
Mack analytic estimate	0	6	11	17	20
Min DCL and BDCL	0	12	18	25	25

| MSEP^(1/2) / BE |	< 25%	< 30%	< 40%	< 50%	≥ 50%
CL bootstrap	23	26	27	28	2
Mack analytic estimate	20	25	26	26	4
Min DCL and BDCL	25	26	26	26	4

Results: coefficient of variation (incremental figures)

- **Number of triangles where the difference (in % of best estimate) between CL bootstrap (by England) and considered alternatives falls in the range:**

	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
Mack analytic estimate	16	9	3	1	1
DCL, "cut-off"	11	3	5	3	8
DCL, "nullifying and rescaling"	21	3	1	1	4
BDCL, "cut-off"	8	4	7	3	8
BDCL, "nullifying and rescaling"	19	5	1	0	5

- Volatility of estimates similar as for the DCL method
  - As the best estimate may be substantially different, so may be the standard deviation

Results: VaR 95% and VaR 99% (incremental figures)

- **Number of triangles where the difference between CL and DCL bootstraps (in % of best estimate) falls in the range:**

VaR 95%	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, "cut-off"	12	2	6	0	10
DCL, "nullifying and rescaling"	16	6	2	1	5
BDCL, "cut-off"	6	7	7	0	10
BDCL, "nullifying and rescaling"	11	8	5	1	5

VaR 99%	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, "cut-off"	10	3	0	5	12
DCL, "nullifying and rescaling"	13	7	4	0	6
BDCL, "cut-off"	4	3	3	8	12
BDCL, "nullifying and rescaling"	11	6	5	2	6

Results: VaR 95% and VaR 99% (incremental figures)

- Again, adjustments in (B)DCL payment delay parameters ("cut-off" or "nullifying and rescaling") led in several cases to unrealistic estimates, but such an estimate was rarely seen for both adjusting methods at once
  - If we take the estimate "closer" to the chain-ladder bootstrap

VaR 95%	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, choosing closer est.	17	7	2	0	4
BDCL, choosing closer est.	11	11	4	0	4

VaR 99%	< 5%	5%-10%	10%-25%	25%-50%	≥ 50%
DCL, choosing closer est.	16	6	3	1	4
BDCL, choosing closer est.	11	7	6	2	4

External data in DCL method

Zero claims
Claims development inflation

Double Chain Ladder, Claims Development Inflation and Zero Claims

M.D. Martínez-Miranda, J. P. Nielsen,

R. Verrall, M. Wüthrich

August 2013

Adding prior knowledge

Adding prior knowledge to the (B)DCL

- About future zero claims
- About future severity development inflation

The aim is to improve the bootstrap, not the best estimate

Adding prior knowledge

Parametric model structure

$\Delta_m = (X_{ij} : 1 \leq i+j \leq m)$ **triangle of claims paid**

$\aleph_m = (N_{ij} : 1 \leq i+j \leq m)$ **triangle of incurred claims**

N_{ijl}^{paid} **– number of claims payment originating from the reported N_{ij} claims which are paid with l periods delay after being reported,** $l = 0, \ldots, m-1$**;**

$Y_{ijl}(k)$ **– individual payments which arise from N_{ijl}^{paid}. For total payments Xij we have**

$$X_{ij} = \sum_{l=0}^{j} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}$$

Assumptions

- N_{ij} independent, with Poisson distribution (as in Mack chain-ladder)
- Given N_{ij}, the numbers of payments follow a multinomial distribution

$$(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \text{Multi}(N_{ij}; p_0, \ldots, p_d)$$

- $Y_{ijl}(k)$ i.i.d., mutually independent, independent of N_{ij}, having a mixed type distribution
  - With probability Q_i of being zero claim
  - Conditionally, if not being a zero claim, $Y_{ijl}(k)$ has a distribution with mean μ_{ij} and variance σ_{ij}^2 where
    - $\mu_{ij} = \mu\gamma_i\delta_j$
    - $\sigma_{ij}^2 = \sigma^2\gamma_i^2\delta_i^2$
    - δ_j and γ_i can be interpreted as an inflation in the payment and accident periods

Treating only the claims development inflation is easy (i.e. we assume no zero claims, $Q_i = 0$)

- We simply adjust the paid triangle with values

$$\widetilde{X}_{ij} = X_{ij}/\delta_j$$

- Then the adjusted paid triangle together with the triangle of counts follow the same model assumptions with $\delta_j = 1$ which means that the usual DCL algorithm can be applied.
- The DCL algorithm provides the prediction

$$\widetilde{X}_{ij}^{DCL}$$

- And, finally, this prediction is transformed back by

$$\widetilde{X}_{ij}^{DCLP} = \delta_j \widetilde{X}_{ij}^{DCL}$$

Incorporation of both zero claims and claims development inflation is based on a similar adjustment

$$\widetilde{X}_{ij} = X_{ij}/[\delta_j(1 - Q_i)]$$

but the rest is more complicated.

We illustrate how the bootstrap procedure changes in this case.

Adding prior knowledge

Bootstrap modification incorporating prior knowledge

RBNS part of the algorithm

1. **From the triangles of counts and payments adjusted using the formula on the previous slide, derive "DCL" parameters:** p_l, γ_i, μ, σ^2.
2. **Generate new triangle of payments as follows**
   - Simulate the payments N_{ijl}^{paid*} from a multinomial distribution $(N_{ij}; p_0, \ldots, p_{m-1})$
   - Simulate the number of nonzero payments N_{ij}^{paid*} using a binomial distribution with the size $\Sigma\, N_{ijl}^{paid*}$ and the probability $(1\text{-}Q_i)$
   - Simulate the payments X_{ij}^* from gamma distribution with the shape parameter $N_{ij}^{paid*}\gamma_i\mu/\sigma_i$ and the scale parameter $\sigma_i^2/\gamma_i\mu$, where $\sigma_i^2 = (1\text{-}Q_i)\sigma^2 - Q_i\mu$
3. **Estimate "DCL" parameters again based on the original triangle of claims counts and simulated triangle of payments which is again adjusted for zero claims:** $X_{ij}^{*,adj} = X_{ij}^* / (1\text{-}Q_i)$.
4. **Simulate the RBNS cash-flows using the step 2 with parameters obtained in the step 3. Adjust the simulated payments by a multiplication with the inflation factor** δ_j.
5. **Repeat the steps 2-4 *B* times (Monte Carlo).**

IBNR part of the algorithm is modified in a similar way.

“Triple” chain-ladder

Statistical modelling and forecasting of outstanding liabilities in non-life insurance

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M. Wüthrich

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Main differences to double chain-ladder

- The third triangle, *number of payments*, is assumed to be available
  - It is used to model more detailed payment pattern
  - In particular, parametric model can deal with more than one payment per claim
  - More detailed model vs. more complex
- More general assumptions, namely concerning inflation
  - Calendar year inflation is not modeled. It is assumed to be extracted up front, if necessary

$\Delta_m = \{X_{ij} : (i,j) \in \mathcal{I}\}$ **triangle of claims paid**

$\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}\}$ **triangle of incurred claims counts**

$R_m = \{N_{ij}^{paid} : (i,j) \in \mathcal{I}\}$ **triangle of number of payments**

N_{ijl}^{paid} – number of payments from N_{ij} claims with l periods delay after being reported, $l = 0, \ldots, m\text{-}1$;

N_{ij}^{paid} – number of claims incurred in period i and (fully) paid with j periods delay

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + N_{i,j-2,2}^{paid} + \ldots + N_{i,0,j}^{paid}$$

$Y_{ijl}(k)$ – individual claim payments, $k = 1, \ldots, N_{ijl}^{paid}$. Hence,

$$X_{ij} = \sum_{l=0}^{j} X_{ijl}, \qquad X_{ijl} = \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}(k)$$

Assumptions

- All random variables in different accident years are independent
- N_{ij} are independent and Poisson distributed with $E[N_{ij}] = \vartheta_i \beta_j$ and identification $\vartheta_1 = 1$
- Claims payments X_{ijl} are, conditionally given number of $N_{i0}, \ldots, N_{i,m-1}$, independent (in l) and compound Poisson distributed with
  - $N_{ijl}^{paid} \mid \{N_{i0}, \ldots, N_{i,m-1}\} \sim \text{Poi}(N_{ij}\pi_l)$ with given parameter $\pi_l > 0$;
  - $Y_{ijl}(k) \mid \{N_{i0}, \ldots, N_{i,m-1}\}$ are i.i.d. (in k) with $E[Y_{ijl}(k)] = \nu_i \mu_{jl}$, $E[Y_{ijl}(k)^2] = \nu_i^2 s_{jl}^2$ with given parameters $\nu_i, \mu_{jl}, s_{jl} > 0$, $\nu_1 = 1$

Parameter estimation procedure

- Applying classical chain-ladder algorithm on three triangles gives estimates of ϑ_i, β_j, π_l, ν_i, μ_{jl}
  - Only first moment assumptions are necessary to justify the procedure
  - Enough to provide best estimate
  - Note also that the model is over-parametrized: authors suggests to put either $\mu_{jl} = \mu_j$ or $\mu_{jl} = \mu_l$ and, consistently, $s_{jl} = s_j$ or $s_{jl} = s_l$
- Estimator for the second moment parameters s_{jl} along with further distributional assumptions
  - Necessary for the bootstrap
  - Rely on the sample estimator

From the assumptions, it follows that

- $\mathrm{E}[N_{ij}] = \vartheta_i \beta_j$
- $\mathrm{E}\left[N_{ij}^{paid}\right] = \vartheta_i \lambda_j$, where $\lambda_j = \sum_{l=0}^{j} \beta_{j-l} \pi_l$
- $\mathrm{E}[X_{ij}] = \alpha_i \gamma_j$, where $\alpha_i = \vartheta_i \upsilon_i$ and $\gamma_j = \sum_{l=0}^{j} \beta_{j-l} \pi_l \mu_{j-l,l}$

and

- $\mathrm{Var}[N_{ij}] = \vartheta_i \beta_j$
- $\mathrm{Var}\left[N_{ij}^{paid}\right] = \vartheta_i t_j^2$, where $t_j^2 = \sum_{l=0}^{j} \beta_{j-l} \pi_l (1 + \pi_l)$
- $\mathrm{Var}[X_{ij}] = \alpha_i \nu_i \sigma_j^2$, where $\sigma_j^2 = \sum_{l=0}^{j} \beta_{j-l} \pi_l \mu_{j-l,l} \left(\pi_l \mu_{j-l,l} + \frac{s_{j-l,l}^2}{\mu_{j-l,l}} \right)$

Thus

- All three random variables – number of claims, number of payments, claim payments – follow (over-dispersed) Poisson model

➔ Ordinary chain-ladder method provides maximum-likelihood estimators of parameters

Ordinary chain-ladder provides by solving system of linear equations (see example for $\aleph_m$) the estimates of:

$\hat{\vartheta}_i^{(1)}$ and $\hat{\beta}_j$ from triangle $\aleph_m$

$\hat{\vartheta}_i^{(2)}$ and $\hat{\lambda}_j$ from triangle R_m

$\hat{\alpha}_i$ and $\hat{\gamma}_j$ from triangle Δ_m

$$\sum_{k=0}^{m-i} \mathbb{E}[N_{i,k}] = \vartheta_i \sum_{k=0}^{m-i} \beta_k \qquad \text{for } i = 1, \ldots, m,$$

$$\sum_{k=1}^{m-j} \mathbb{E}[N_{k,j}] = \beta_j \sum_{k=1}^{m-j} \vartheta_k \qquad \text{for } j = 0, \ldots, m-1$$

Chosen initialization $\vartheta_1^{(1)} = \vartheta_1^{(2)} = \alpha_1 = 1$ makes solving of the linear system easier. Note that expected values are replaced with observed values. Thus, in the example:

- The first equation for $i = 1$ provides the sum $\beta_0+\beta_1+\ldots+\beta_{m-1}$
- The second equation for $j = m-1$ provides solution for β_{m-1}
- The first equation then for $i = 2$ provides solution for ϑ_2, etc.

There are two estimates of ϑ

- Should not be too different (otherwise it is an indication that the model does not fit)
- It is proposed to use average as the estimate

From obtained estimates, we can derive estimates of

- Inflation parameters v_i from the equation $\alpha_i = \vartheta_i v_i,\ i = 1, \dots, m$
- Payment pattern parameters π_l from the linear system $\lambda_j = \sum_{l=0}^{j} \beta_{j-l}\pi_l,\ j = 0, \dots, m-1$

Due to the above-mentioned over-parametrisation of the model, it is suggested to assume

- either $\mu_{jl} = \mu_j$ – then $\gamma_j = \sum_{l=0}^{j} \beta_{j-l}\pi_l\mu_{j-l,l} = \sum_{l=0}^{j} \beta_{j-l}\pi_l\mu_{j-l}$
- or $\mu_{jl} = \mu_l$ – then $\gamma_j = \sum_{l=0}^{j} \beta_{j-l}\pi_l\mu_{j-l,l} = \sum_{l=0}^{j} \beta_{j-l}\pi_l\mu_l$

In both cases, the parameters can be estimated again by solving the respective linear system

Conditionally expected outstanding loss liability

$$Z_m = \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} E\left[X_{ij} | \aleph_m, R_m, \Delta_m\right]$$

Using the model setting, this can be wrriten as

$$Z_m = \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{j} E\left[\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)} | \aleph_m, R_m, \Delta_m\right]$$
$$= \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=i+j-m,}^{j} E\left[\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)} | \aleph_m, R_m, \Delta_m\right] + \sum_{i=2}^{m} \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{i+j-m-1} E\left[\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)} | \aleph_m, R_m, \Delta_m\right]$$

The decoupling is done in a way that

- The first part corresponds to already reported claims (leading to the estimate of RBNS)
- The second part corresponds to not yet reported claims (leading to the estimate of IBNR)

Using the model assumptions, we obtain

$$Z_m^{\mathrm{RBNS}} = \sum_{i=2}^{m} \nu_i \sum_{j=m-i+1}^{m-1} \sum_{l=i+j-m}^{j} N_{i,j-l}\, \pi_l\, \mu_{j-l,l},$$

$$Z_m^{\mathrm{IBNR}} = \sum_{i=2}^{m} \vartheta_i\, \nu_i \sum_{j=m-i+1}^{m-1} \sum_{l=0}^{i+j-m-1} \beta_{j-l}\, \pi_l\, \nu_i\, \mu_{j-l,l}$$

Again, the ordinary chain-ladder estimate is obtained if the claim numbers in the "RBNS" part are replaced by

$$N_{i,j} = \widehat{\vartheta}_i \widehat{\beta}_j.$$

- Unlike for the ordinary chain-ladder, we have a separate estimate for RBNS and IBNR

Notes

- It feels more natural to use observed values N_{ij} than to replace them as above
- The estimators above do not include the tail. The tail can be estimated similarly to the double chain-ladder. However, the estimate again relies on two assumptions: that no further claims will be reported after m-1 periods and (as the whole method) that the payment pattern is not longer than m-1 periods.

Triple chain-ladder
Case study – best estimate

a.y. i	$\widehat{Z}_m^{\text{RBNS}+}$	$\widehat{Z}_m^{\text{IBNR}+}$	$\widehat{Z}_m^{+}$	$\widehat{Z}_m^{CL}$	difference	in %
1	536	0	536		536	
2	1 540	0	1 540	0	1 540	
3	23 799	0	23 799	2 220	21 579	971.8%
4	162 275	0	162 275	147 434	14 841	10.1%
5	291 122	790	291 912	280 056	11 855	4.2%
6	415 955	1 590	417 545	408 154	9 391	2.3%
7	584 991	3 300	588 291	569 060	19 231	3.4%
8	605 767	3 676	609 443	583 785	25 658	4.4%
9	704 687	5 039	709 726	675 363	34 363	5.1%
10	803 884	6 343	810 228	764 373	45 855	6.0%
11	1 054 124	10 037	1 064 161	1 004 331	59 829	6.0%
12	1 397 607	22 068	1 419 675	1 352 819	66 856	4.9%
13	1 999 243	84 680	2 083 922	2 076 674	7 248	0.3%
14	4 221 084	1 474 793	5 695 877	5 487 650	208 227	3.8%
total	12 266 615	1 612 315	13 878 930	13 351 921	527 009	3.9%

The bootstrap procedure is analogous to the one for double chain-ladder

- Parametric bootstrap
  - Need to specify the distribution of claim payments (gamma chosen in the paper)
  - Need to estimate variance parameters in order to estimate the shape and the scale parameter of this distribution

Estimate of the variance parameters

- To avoid over-parametrisation, it is suggested to put either $s_{jl} = s_j$ or $s_{jl} = s_l$. We consider the second case here.
- We have $\mathrm{E}[X_{ij}] = \alpha_i \gamma_j$ and $\mathrm{Var}[X_{ij}] = \alpha_i \nu_i \sigma_j^2$ which implies

$$\mathbb{E}\left[\frac{X_{i,j} - \alpha_i \gamma_j}{\sqrt{\alpha_i \nu_i}}\right] = 0 \qquad \text{and} \qquad \mathrm{Var}\left(\frac{X_{i,j} - \alpha_i \gamma_j}{\sqrt{\alpha_i \nu_i}}\right) = \sigma_j^2$$

- The sampler estimator then provides for $j = 0, \ldots, m\text{-}2$ (it is suggested to put $\hat{\sigma}_{m-1}^2 = \hat{\sigma}_{m-2}^2$)

$$\widehat{\sigma}_j^2 = \frac{1}{m-j-1} \sum_{i=1}^{m-j} \left(\frac{X_{i,j} - \widehat{\alpha}_i \widehat{\gamma}_j}{\sqrt{\widehat{\alpha}_i \widehat{\nu}_i}}\right)$$

- The formula $\sigma_j^2 = \sum_{l=0}^{j} \beta_{j-l} \pi_l \mu_{j-l,l} \left(\pi_l \mu_{j-l,l} + \frac{s_{j-l,l}^2}{\mu_{j-l,l}}\right) = \sum_{l=0}^{j} \beta_{j-l} \pi_l \mu_l \left(\pi_l \mu_l + \frac{s_l^2}{\mu_l}\right)$ determines then a linear system for s_l^2
- It is suggested to check coefficients of variation $\widehat{\mathrm{vco}} = \hat{s}_l / \hat{\mu}_l$ for their plausibility

Proposed algorithm for the bootstrapping procedure – RBNS part

Estimate of process variance only – do only steps 1, 4 and 5 (using parameters estimated in the step 1).

1. Parameters estimation

- Apply the triple chain-ladder procedure to estimate all model parameters ($Y_{ijl} \sim \Gamma(\hat{\lambda} = \frac{\hat{\mu}_l^2}{\hat{s}_l^2 - \hat{\mu}_l^2}, \hat{\kappa} = \frac{(\hat{s}_l^2 - \hat{\mu}_l^2)\hat{v}_i}{\hat{\mu}_l})$)

2. Bootstrapping the data

- Keep the same counts N, but bootstrap the aggregate number of payments R* and payments X* as follows
  - Simulate the payment delay: $N_{ijl}^{paid*} \sim \mathrm{Poi}(N_{ij}\hat{\pi}_l)$ with $\hat{\pi}_l$ estimated in the step 1
  - Simulate the aggregate payments X_{ij}^* using simulated N_{ijl}^{paid*} and gamma distribution estimated in the step 1

3. Bootstrapping the parameters

- From the bootstrap data (N, R^*, X^*) generated at step 2 obtain new estimates for parameters

4. Bootstrapping the RBNS prediction

- Simulate the delay as in the step 2
- Simulate the aggregate payments as in the step 2
- Get the bootstrapped RBNS prediction

5. Monte Carlo approximation

- Repeat steps 2-4 *B* times and get the empirical bootstrap distribution of the RBNS part of the reserve

Proposed algorithm for the bootstrapping procedure – IBNR part

- Analogous to RBNS part
- Steps 2-4 include the estimation and the simulation of the number of reported claims in the lower triangle
  - Number of reported claims are simulated from the Poisson distribution with means $\hat{\vartheta}_i\hat{\beta}_j$

Case study

- Higher uncertainty compared to DCL
- The advantage of DCL (less parameters) is not outweighted by the more detailed modeling in TCL

	TCL			DCL		
	RBNS	IBNR	total	RBNS	IBNR	total
mean	12 312 055	1 683 054	13 995 109	11 758 152	1 585 151	13 343 303
$MSEP^{1/2}$	2 273 294	326 382	2 324 966	1 881 154	485 312	2 018 112
1%	8 090 717	1 131 376	9 615 040	8 081 739	687 623	9 314 398
5%	9 088 207	1 262 754	10 685 634	9 012 040	897 886	10 408 658
50%	12 040 963	1 640 097	13 723 567	11 637 796	1 532 079	13 243 493
95%	16 325 473	2 222 831	18 101 695	14 869 197	2 448 915	16 729 435
99%	18 860 539	2 709 200	20 660 941	16 516 558	2 941 469	18 487 830

Summary

Summary

Double-chain ladder

- Use primarily information from two triangles: *claims counts* and *paid*
- Two formulations for best estimate
  - First moment (distribution-free)
  - Parametric
- Can replicate classical chain-ladder results
  - with a split of the RBNS and IBNR cash-flows
- Provides an alternative estimate *which is more natural to the underlying assumptions*
- Parametric bootstrapping can be used for an assessment of the full distribution
  - Alternative to the bootstrapping procedures common for classical chain-ladder
- Allows for several extensions (BDCL, Triple chain-ladder)
- Prior knowledge can be incorporated in the assessment of best estimate and in the bootstrap procedure
  - Zero claims
  - Future claims development inflation

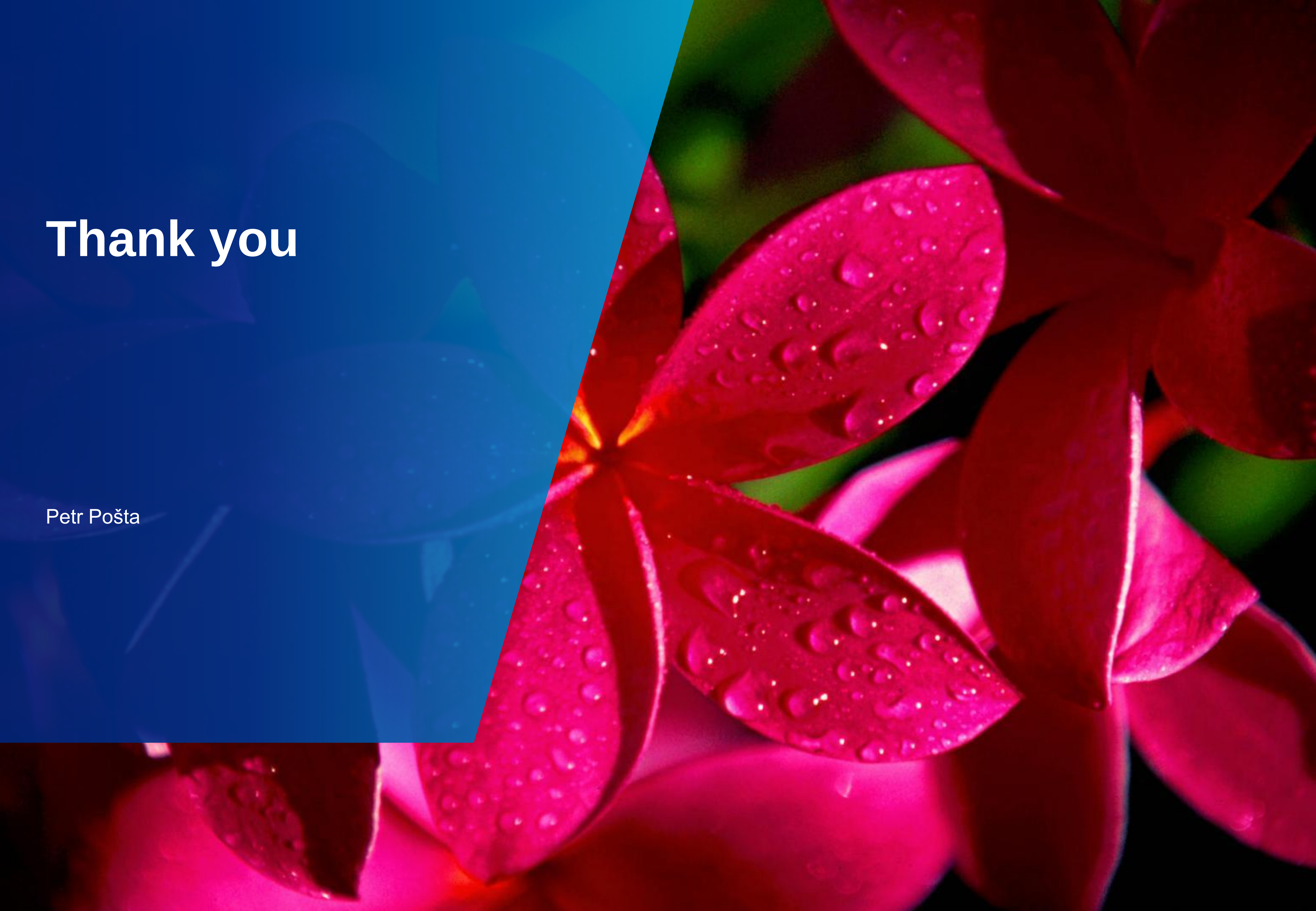
Implementation of DCL and BDCL in R is publicly available

Questions

&

Comments

?



Thank you

Petr Pošta

cutting through complexity

Mean

$$
\begin{aligned}
E[X_{ij}|\aleph_m] &= E[E[X_{ij}|N_{ij}^{paid}]|\aleph_m] \\
&= E\left[E\left[\sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}|N_{ij}^{paid}\right]|\aleph_m\right] \\
&= E[N_{ij}^{paid}E[Y_{ij}^{(k)}]|\aleph_m] \\
&= E[N_{ij}^{paid}|\aleph_m]E[Y_{ij}^{(k)}]
\end{aligned}
$$

Variance

$$
\begin{aligned}
V[X_{ij}|\aleph_m] &= E[V[X_{ij}|N_{ij}^{paid}]|\aleph_m] + V[E[X_{ij}|N_{ij}^{paid}]|\aleph_m] \\
&= E[V[\sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}|N_{ij}^{paid}]|\aleph_m] + V[N_{ij}^{paid}E[Y_{ij}^{(k)}]|\aleph_m] \\
&= E[N_{ij}^{paid}V[Y_{ij}^{(k)}]|\aleph_m] + V[N_{ij}^{paid}E[Y_{ij}^{(k)}]|\aleph_m]
\end{aligned}
$$

Since we assume (without any loss of generality, we omit indeces *i*)

$$E[Y_{ij}(k)] = \mu, \qquad V[Y_{ij}(k)] = \sigma^2$$

Thus

$$E[X_{ij}|\aleph_m] = E[N_{ij}^{paid}|\aleph_m]\mu$$

$$V[X_{ij}|\aleph_m] = E[N_{ij}^{paid}|\aleph_m]\sigma^2 + V[N_{ij}^{paid}|\aleph_m]\mu^2$$

Using the assumption of conditional multinomial distribution of N_{ij}^{paid}

$$\begin{aligned} E[N_{ij}^{paid}|\aleph_m] &= E\left[\sum_{k=0}^{\min\{j,d\}} N_{i,j-k,k}^{paid}|\aleph_m\right] \\ &= \sum_{k=0}^{\min\{j,d\}} E[N_{i,j-k,k}^{paid}|\aleph_m] \\ &= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k}p_k \end{aligned}$$

Assuming that the numbers of claims paid from different origin years are uncorrelated

$$\begin{aligned} V[N_{ij}^{paid}|\aleph_m] &= V\left[\sum_{k=0}^{\min\{j,d\}} N_{i,j-k,k}^{paid}|\aleph_m\right] \\ &= \sum_{k=0}^{\min\{j,d\}} V[N_{i,j-k,k}^{paid}|\aleph_m] \\ &= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k}p_k(1-p_k) \end{aligned}$$

Hence

$$E[X_{ij}|\aleph_m] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \mu$$

$$\begin{aligned} V[X_{ij}|\aleph_m] &= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \sigma^2 + \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (1-p_k)\mu^2 \\ &= \sum_{k=0}^{\min(j,d)} N_{i,j-k} \{\sigma^2 p_k + \mu^2 p_k (1-p_k)\} \\ &\approx \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (\sigma^2 + \mu^2) \end{aligned}$$

Last approximation is done so that the variance is proportional to the mean

➔ An over-dispersed Poisson model may be used.