Advanced Claims Reserving Methods

Michal Gerthofer

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Introduction

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Outline

Advanced Claims Reserving Methods - Motivation

- **Complex reserving problem** in non-life insurance
- Claims reserve is the **biggest balance sheet item** for non-life insurance company
- Utilization of the **micro data** within insurance
- **Regulatory framework**
    - Stochastic modeling
    - Utilization of available data within the company

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Outline

Advanced Claims Reserving Methods - Objective

- Chain ladder - **distribution free** approach $\rightsquigarrow$ **does not provide distributional properties**
  - Chain ladder Bootstrap - **mimics unknown distribution**
  - **Watch out!** There is also **second stage of the Bootstrap** - simulation of the process error. Distribution must be defined anyway
- **Stochastic models** based on aggregated data able to cope with **dependent variables**
  - Generalized Estimation Equation
  - Generalized Linear Mixed Models
- Development of the **micro-model**
  - Utilization of **all data about the individual claims**
  - Increased **precision of prediction**
  - **Machine learning methods** used
    - Neural networks
    - Hurdle models

Outline

Linear Mixed Models

Let's assume that Y_{it} follows Linear Mixed model

$$Y_{it} = \mathbf{X}_{it}{}^{T}\beta + u_{it}$$

- The breakup of residuals:

$$u_{it} = b_i + \varepsilon_{it}$$

- In a model with random effects, simple correlation structure is introduced:

$$\begin{gathered}
Y_{it} = \mathbf{X}_{it}{}^{T}\beta + b_i + \varepsilon_{it}, \\
b_i \sim iid\left(0, \sigma_b^2 > 0\right), \\
\varepsilon_{it} \sim iid\left(0, \sigma_\varepsilon^2 > 0\right), \\
E\left(\varepsilon_{it} b_j\right) = 0, \quad \forall\ i, j \ and\ t, \qquad E\left(b_i b_j\right) = 0, \quad i \neq j
\end{gathered}$$

Linear Mixed Models

- Introducing the correlation:

$$\begin{aligned} Cov\left(u_{it}, u_{is}\right) &= Cov\left(\varepsilon_{it} + b_i, \varepsilon_{is} + b_i\right) \\ &= Var\left(b_i\right) + Cov\left(\varepsilon_{it}, \varepsilon_{is}\right) = \\ &= \begin{cases} \sigma_b^2 & t \neq s, \ \forall i, \\ \sigma_b^2 + \sigma_\varepsilon^2 & t = s, \ \forall i. \end{cases} \end{aligned}$$

$$Corr\left(u_{it}, u_{js}\right) = \begin{cases} \frac{\sigma_b^2}{\sigma_b^2 + \sigma_\varepsilon^2} & t \neq s, \ i = j, \\ 1 & t = s, \ i = j, \\ 0 & \text{otherwise} \end{cases}$$

- LMM (in general):

$$Y_{it} = \mathbf{X}_{it}{}^T \beta + \mathbf{Z}_{it}{}^T \mathbf{b}_i + \varepsilon_{it}$$

Bayes Approach in Linear Mixed Models

- Prediction of random variable translates into the problem of predicting the conditional mean of $\mathbf{b}_i$, given the vector of response $\mathbf{Y}_i$. Using properties of joint multivariate normal distribution of $\mathbf{b}_i$ and ε_i, it can be written as $E\left(\mathbf{b}_i|\mathbf{Y}_i\right) = \mathbf{G}\mathbf{Z}_i^T\mathbf{\Sigma}_i^{-1}\left(\mathbf{Y}_i - \mathbf{X}_i\widehat{\beta}\right)$, where $Cov\left(\mathbf{Y}_i\right) = \mathbf{Z}_i\mathbf{G}\mathbf{Z}_i^T + \mathbf{R}_i$

$$\widehat{\mathbf{Y}}_i = \mathbf{X}_i\widehat{\beta} + \mathbf{Z}_i\widehat{\mathbf{b}}_i, \widehat{\mathbf{Y}}_i = \left(\widehat{\boldsymbol{R}}_i\widehat{\mathbf{\Sigma}}_i^{-1}\right)\mathbf{X}_i$$

$$\widehat{\mathbf{Y}}_i = \left(\widehat{\boldsymbol{R}}_i\widehat{\mathbf{\Sigma}}_i^{-1}\right)\mathbf{X}_i\widehat{\beta} + \left(\mathbf{I}_{n_i} - \widehat{\boldsymbol{R}}_i\widehat{\mathbf{\Sigma}}_i^{-1}\right)\mathbf{Y}_i,$$

- where $\mathbf{R}_i = Cov(\varepsilon_i) = \sigma_\varepsilon^2\mathbf{I}_{n_i}$, $\mathbf{G} = Cov(\mathbf{b}_i)$ and $\mathbf{\Sigma}_i = Cov(\mathbf{Y}_i)$
- If the within-subject variability $\mathbf{R}_i$ is relatively large compared to the between-subject variability $\mathbf{\Sigma}_i$, more weight is given to $\mathbf{X}_i\widehat{\beta}$ than to the i-th observed response

Generalization of Linear Mixed Models

- Given b_i, components of $\mathbf{Y}_i$ are conditionally independent, with density belonging to the **exponential family of distributions**:
$$f(y_{it}|\mathbf{b}_i) = \exp\left\{\frac{Y_{it}\theta_{it} - b(\theta_{it})}{\varphi} + c(Y_{it},\ \varphi)\right\}$$
- Then the **conditional mean** of Y_{it} given $\mathbf{b}_i$ is
$$\mu_{it} \equiv E(Y_{it}|\mathbf{b}_i) = b'(\theta_{it})$$
and the **conditional variance** of Y_{it} given $\mathbf{b}_i$ has the following form:
$$Var(Y_{it}|\mathbf{b}_i) = \varphi b''(\theta_{it}) \equiv \varphi V(\mu_{it})$$
- Furthermore, it is assumed that μ_{it} is related to the **linear predictor**
$$\eta_{it} = \mathbf{X}_{it}{}^T\boldsymbol{\beta} + \mathbf{Z}_{it}^T\boldsymbol{b}_i$$
through the **link function** $g(\mu_{it}) = \eta_{it}$
- Conditional Y_{it} given $\mathbf{b}_i$ satisfies the GLM and the inclusion of $\mathbf{b}_i$ in all η_{it} **brings in correlation** between $Y_{i1}, \ldots, Y_{in_i}$

Outline

Basics of Generalized Estimation Equations

- The main idea behind GEE is to generalize and extend the usual likelihood equations from GLM by including the **covariance matrix** of the vector $\mathbf{Y}$
- The biggest advantage of this model:
  - **No need to specify the whole distribution** of the response
- On the other hand, the following have to be defined:
  - The mean structure
  - The **mean-variance relationship**
  - Specification of the covariance structure
- The first two conditions are similar to the GLM

Definition of Generalized Estimation Equations

- Unbalanced design with independence between individuals $\mathbf{Y}_i,\ i = 1, 2, \ldots,\ N$ is assumed like in GLMM.
- Denote **expected value** of response

$$\mu_{it} \equiv E(Y_{it}),$$

which depends on covariates, $\mathbf{X}_{it}$ as follows

$$g(\mu_{it}) = \eta_{it} = \mathbf{X}_{it}^T \boldsymbol{\beta}$$

- It is also assumed that the variance of each Y_{it} depends on the mean according to

$$Var(Y_{it}) = \varphi V(\mu_{it}),$$

where $V(\cdot)$ is a **known variance function** and $\varphi > 0$ is a **scale or dispersion parameter**, that can be known or may need to be estimated

Impact of Correlation Matrix

- Furthermore, correlation between components of $\mathbf{Y}_i$ is represented by a ***working correlation matrix***

$$\mathbf{C}_i \equiv \mathbf{C}_i(\boldsymbol{\alpha}),$$

  where $\boldsymbol{\alpha}$ is $s \times 1$ vector of unknown parameters
- The name "working" comes from the fact that the **structure of $\mathbf{C}_i$ does not need to be correctly specified and asymptotic properties of estimate still hold**
- The corresponding working covariance matrix for i-th subject can be constructed as the product of standard deviations and working correlation matrix

$$\mathbf{V}_i = \varphi \mathbf{A}_i^{1/2} \mathbf{C}_i(\boldsymbol{\alpha}) \mathbf{A}_i^{1/2},$$

  where $\mathbf{A}_i$ is diagonal matrix with $V(\mu_{it})$ on the diagonal

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Claims Reserving using Generalized Linear Mixed Models

- Due to interpretation and useful properties **log link function** was chosen
- Three various **distribution functions**:
  - Gamma
  - Poisson
  - Negative binomial
- **Linear predictor**:

$$log[E(Y_{it}|b_i)] = log(\mu_{it}) = \eta_{it} = \beta_0 + b_i + \beta_t,$$

  - β_0 is intercept
  - b_i is random effect (**Gaussian distribution with zero mean**)
  - β_t captures the impact of change for particular development year
  - b_i is not estimated but **predicted**

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Claims Reserving using Generalized Estimation Equations

- **Log link** used as well as in GLMM
- **Variance function**:

$$V(\mu_{it}) = \begin{cases} 1, \\ \mu_{it}, \\ \mu_{it}^2 \end{cases}$$

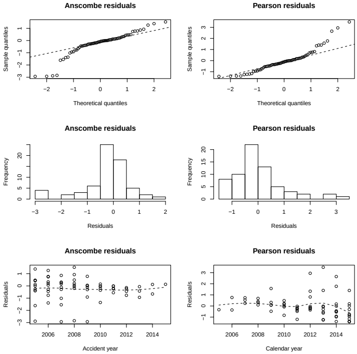
- **Working correlation matrix**: independent, AR(1) and exchangeable
  - Low number of parameters
- **Linear predictor**:

$$log[E(Y_{it})] = log(\mu_{it}) = \eta_{it} = \gamma + \alpha_i + \beta_t,$$

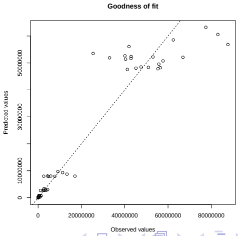
  - $\alpha_1 = \beta_1 = 0$ and α_i represents effect of accident year i, β_t effect of development year t

Residual Diagnostic of Gamma Model

- Based on residual diagnostic **Gamma model** was chosen
- **Predictions** of the models was further **compared with new diagonal** see Table below



Distribution	Gamma	Poison	Negative binomial
RBNS + IBNR (ths.)	84 926	91 485	88 706
Diff. real./est. diag (ths.)	-37 721	-44 245	-41 875
MSEP (bil.)	866	1 466	1 297

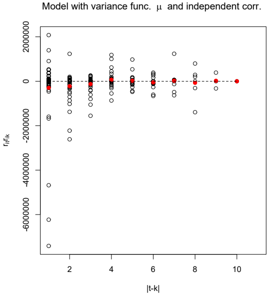


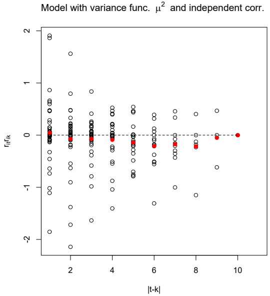
Residual Diagnostic for Generalized Estimation Equations

- Choice of **working correlation structure**
  - If the mean structure is correct, the following should hold for Pearson residuals

$$
\begin{aligned}
r_{i,t} &\approx 0 \\
Var(r_{i,t}) &\approx \varphi \\
r_{i,t}r_{i,k} &\approx \varphi \{\mathbf{C}_i\}_{t,k}, \quad i = 1, 2, \ldots, N, \quad t \neq k \in \{1, 2, \ldots, n_i\}
\end{aligned}
$$

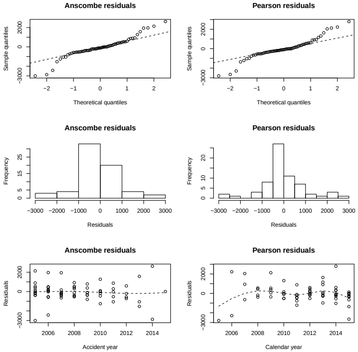
Not possible to clearly identify matrix correlation structure based on Figures below

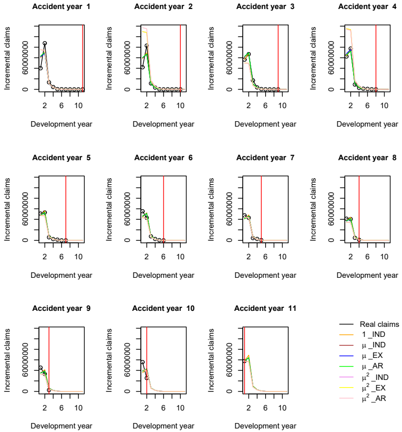




Residual Diagnostic for Generalized Estimation Equations

- Based on residual diagnostic model with **variance function** $\mu_{i,t}$ and **independent correlation structure** is chosen
- Choice of variance function is approved by **QIC**





Comparison of All the Models

- **Predictions** of all the models were further **compared with the new diagonal** see Table below

Model	RBNS + IBNR (ths.)	Diff. diag. (ths.)	MSEP (bil.)
Mack ch. l. (paid triangle)	91 485	-44 245	1 466
GLMM Gamma	84 926	-37 721	866
GLMM Poison	91 485	-44 245	1 466
GLMM Negative binomial	88 706	-41 875	1 297
GEE 1 - IND	97 266	-49 039	1 847
GEE $\mu_{i,t}$ - IND	91 485	-44 245	1 466
GEE $\mu_{i,t}$ - EX	91 762	-44 279	1 466
GEE $\mu_{i,t}$ - AR	91 934	-44 391	1 467
GEE $\mu_{i,t}^2$ - IND	80 919	-35 967	956
GEE $\mu_{i,t}^2$ - EX	80 919	-35 967	956
GEE $\mu_{i,t}^2$ - AR	80 625	-35 793	949

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Conclusions and Discussion

- Chain ladder is **not always the best** approach
  - Necessary to **check assumptions** via residual diagnostic, back-testing
- **Be careful by using incurred triangles** - internal processes may change by the time
- Most of the mentioned stochastic methods are **not able to deal with zeros** in incremental run-off triangle
  - Might be changed with small values but needs further investigation and testing of sensitivity
- Particular GEE or GLMM models **lead to classical GLM** or have **the same predictions as Chain ladder**
- **Powerful generalization of** well known **GLM models** for dependent variables

Outline

Proposed Approach

- Frequency model
  - Predicted number of claims in a given **accident year** i and **reporting delay** t (in years), $N_{i,t}$
  - Consequently, this number will be split into $N_{i,t,c}$, where $N_{i,t} = \sum_{c=1}^{C} N_{i,t,c}$ and C is maximal assumed length of a claim development in years (time from reporting to closing)
  - $N_{i,t,c}$ **represents number of claims in accident year** i, **development year** t **that are settled after** c **years after reporting**
- Severity model
  - j-th **expected yearly payment** $\mathrm{E}Y_{o,t,c,j} = \mu_{t,c,j}$ of claim o will be modeled with respect to reporting delay t and length of claim settlement c, where $j = 1, 2, \ldots, c$
  - Main objective is the modeling of **development patterns** $\boldsymbol{\mu}_{t,c} = (\mu_{t,c,1}, \mu_{t,c,2}, \ldots, \mu_{t,c,c})^T$

Proposed Approach

- Overall reserve R

$$\mathrm{E}R = \sum_{i=1}^{N}\sum_{t=1}^{N}\sum_{c=1}^{C}\mathrm{E}N_{i,t,c}\sum_{j=1}^{c}\mathrm{E}Y_{o,t,c,j}$$

  - This equation holds if $N_{i,t,c}$ is independent with $Y_{o,t,c,j}$
- Estimation of overall reserve

$$\widehat{R} = \sum_{i=1}^{N}\sum_{t=1}^{N}\sum_{c=1}^{C}\widehat{N}_{i,t,c}\sum_{j=1}^{c}\widehat{\mu}_{t,c,j}$$

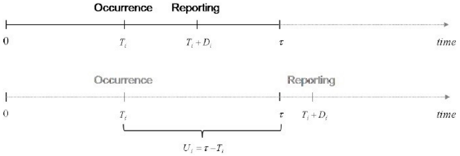
Outline

Dataset

Run-off triangle for incremental number of IBNR claim amounts

Accident year i	Development year j: 1	2	3	4	5	6	7	8	9	10	11	12
2005	**2612**	**617**	**24**	**12**	**0**	**0**	*0*	*0*	*0*	*0*	*0*	*0*
2006	**2405**	**622**	**30**	**8**	**1**	*0*	*0*	*0*	*0*	*0*	*0*	
2007	**2416**	**587**	**41**	**5**	*6*	*0*	*0*	*0*	*0*	*0*		
2008	**2619**	**563**	**42**	*10*	*4*	*1*	*0*	*0*	*0*			
2009	**2081**	**429**	*39*	*8*	*3*	*1*	*0*	*0*				
2010	**1980**	*415*	*61*	*16*	*0*	*0*	*0*					
2011	1745	354	45	13	1	0						
2012	1635	322	27	4	1							
2013	1752	284	17	9								
2014	1705	253	13									
2015	1608	219										
2016	1667											

Maximum Likelihood Estimate



Assumptions

$N_k = \sum_{i=1}^{N} \chi(T_i \in \delta_k)$, $N_k^{(1)} \sim Bi(N_k, p_k)$.
where N_k denotes the number of claims that occurred within the time interval δ_k and p_k is corresponding probability to be reported until the current time τ.

- It is assumed that within this short time interval δ_k, the intensity of the claims process is constant
- In our application δ_k equals to k-th accident week

Maximum Likelihood Estimate

- $N_k^{(1)}$ denotes number of claims reported until the current time τ

Likelihood function

$$l(N_k) = \frac{N_k!}{(N_k - N_k^{(1)})!N_k^{(1)}!} p_k^{N_k^{(1)}} (1 - p_k)^{N_k - N_k^{(1)}}.$$

- Different approach - $N_k^{(1)}$ known, N_k unknown.
- Must estimate p_k, $\forall k$ (distribution of reporting delay)

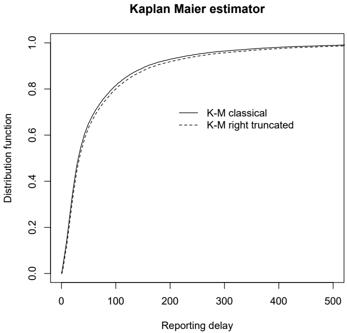
Distribution of Reporting Delay

- Nonparametric estimator
  - Kaplan Maier for truncated data
- Semi-parametric estimator
  - Cox regression for truncated data
- Parametric estimator for truncated data - MLE
  - Weibull
  - Pareto
  - Log-normal

Kaplan Maier for Right Truncated Data

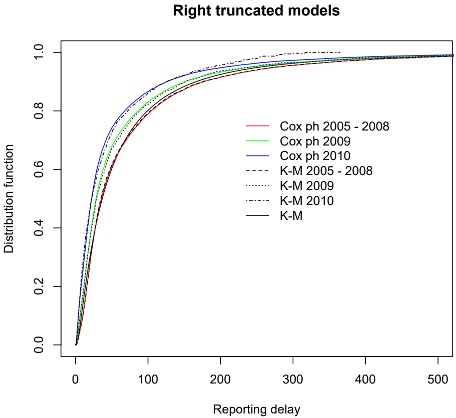
$$\widehat{F}_D^*(d) = \prod_{j:d_{(j)}>d}\left(1-\frac{r_{(j)}}{H_{(j)}}\right),\ r_{(j)} \equiv \sum_{i'=1}^{n'} \chi(d_{i'} = d_{(j)}),\ \ H_{(j)} \equiv \sum_{i'=1}^{n'} \chi(d_{i'} \le d_{(j)} \le u_{i'})$$

where $(u_{i'}, d_{i'})$ are observations of $(U_{i'}, D_{i'})$ and $i' = 1, 2, \ldots, n'$, where n' is the number of observations. Then, $d_{(1)} < d_{(2)} < \ldots < d_{(J)}$ are ordered distinct values of $d_1, d_2, \ldots, d_{n'}$. $F_D^*(d) = \frac{F_D(d)}{F_D(d_{(J)})}$ is just conditional distribution!!!



Cox Regression for Truncated Data

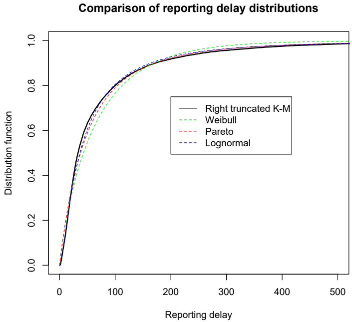
- Assumptions of proportionality holds
- Transformation of variable needed
- All necessary theory can be found in [4]



Parametric Approach - MLE

Estimation

$$\widehat{\boldsymbol{\theta}}_D = \arg\max_{\boldsymbol{\theta}_D} \log \prod_{i'} \frac{f_D(d_{i'}, \boldsymbol{\theta}_D)}{F_D(u_{i'}, \boldsymbol{\theta}_D)},$$



Distribution	Weibull	Pareto	Log-normal
Log-Likelihood	-88523	-87896	-87007

Based on the log-likelihood comparison in the Table above log-normal distribution is preferred

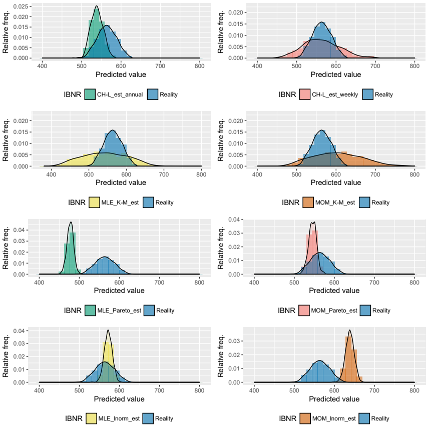
Comparison of Results

- Prediction of IBNR number of claims and comparison with reality

Real	Mack	Pred.	MLE	Pred.	Moment	Pred.
564	Annual	538	K-M classic	447	K-M classic	499
	Weekly	561	K-M truncated	542	K-M truncated	602
			Cox prop. haz.	272	Cox prop. haz.	339
			Weibull	490	Weibull	525
			Pareto	464	Pareto	533
			Log-normal	531	Log-normal	598

- Also **moment method** was used in addition to maximum likelihood
- **Cox regression and classical K-M** are **not very precise**
- It is **not possible to assess prediction of reserve** based on one realization therefore **simulation analysis must be performed**

Simulation Analysis



	Mack annual	Mack weekly	MLE K-M	MLE Pareto	MLE L-nor.	Moment K-M	Moment Pareto	Moment L-nor.
Bias	25	1	21	87	-8	-40	18	-75
MSPE	1422	2712	4526	8084	699	6060	953	6199

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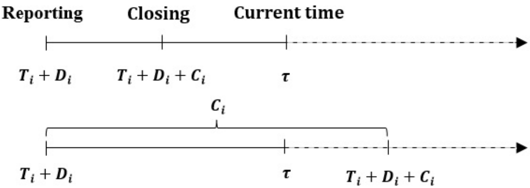
Survival Analysis for Censored Data

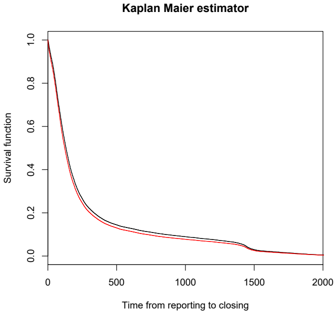


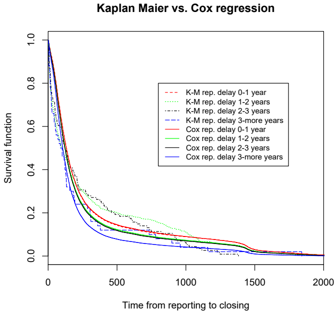
K-M estimation of survival function

$\widehat{S}_C(t) = 1 - \widehat{F}^*_C(t) \sum_{j:c_j > t} \frac{n_j - d_j}{n_j}$,
where k_i is either time from reporting to closing or in case claim hasn't closed yet, censoring time. Further, δ_i is indicator of closed claim and n is the number of observation,
$d_j = \sum_{i=1}^{n} \chi(k_i = c_j, \delta_i = 1)$ represents number of claims at time c_j a $n_j = \sum_{i=1}^{n} \chi(k_i \leq c_j)$ is number of claims in risk at time c_j

Kaplan Maier for Censored Data

- Cox regression not possible to use due to violation of assumptions
- Comparison of Kaplan Maier for censored data and classical Kaplan Maier





Conditional Expected Value

- For claims which are still open the closing time must be predicted. Therefore, conditional expected value $E(T \mid T > t)$ must be predicted in our application.
- Using survival function conditional expected value can be expressed as follows

$$E(T \mid T > t) = t + \int_t^\infty \frac{S(u)}{S(t)} du$$

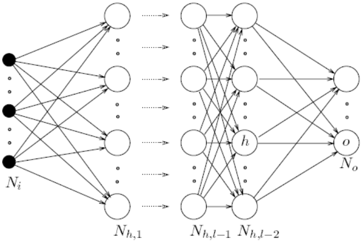
- Estimation of conditional expected value can be expressed using estimated survival function as

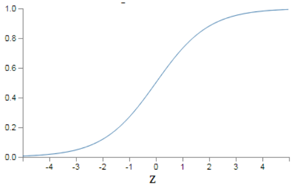
$$E(\widehat{T \mid T} > t) = t + \int_t^\infty \frac{\widehat{S}(u)}{\widehat{S}(t)} du$$

Outline

Structure of Neural Networks

- Hidden layers
  - One
  - Two
- Number of neurons within layers
- Activation function $F\left(\sum_i w_i x_i + b\right)$
  - Sigmoid $F(z) = \frac{1}{1+e^{-z}}$
  - Hyperbolic tangent $F(z) = \tanh(z) = \frac{e^{2z}-1}{e^{2z}+1}$
  - Linear $F(z) = z$
  - ReLu (rectified linear unit) $F(z) = \max(0, z)$





Optimization Algorithm

- **Batch methods** which use the **full training set** to compute the next update to parameters tend to **converge very well**
  - Very few hyper-parameters to tune
  - In practice **computing the cost** and gradient for the entire training set can be very slow and sometimes intractable on a single machine
  - Doesn't give an easy way to incorporate new data in an 'online' setting

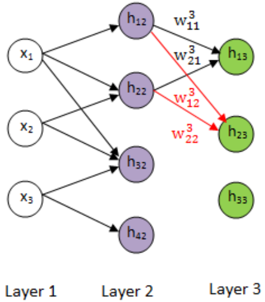
$$\theta = \theta - \gamma \nabla_\theta E\left[L(\theta)\right]$$

  where θ is unknown parameter, γ learning rate and L is loss (cost) function.
- **Stochastic Gradient Descent** (SGD) **addresses all these issues** by following the negative gradient of the objective after seeing only a single or a **few training examples**

$$\theta = \theta - \gamma \nabla_\theta L\left(\theta; x_i, y_i\right)$$

Weights

- θ in SGD denotes all weights w_{jk} and biases b_k for all k and j within neural network
- Upper index w^s_{jk} in figure below denotes hidden layer but it is not necessary



Back Propagation - Generalized Delta Rule

$$L=\sum_{p=1}^{P} L^p, \qquad L^p = \frac{1}{2}\sum_{i=1}^{N_o}(o_i^p - y_i^p)^2, \qquad y_k^p = F(s_k^p) = F\left(\sum_{j=1}^{n} w_{jk}\, y_j^p + b_k\right)$$

where o_i^p is i-th element of p-th observation

- Utilizing **differentiability of the activation function, partial derivation** and **chain rule**

$$\Delta_p w_{j,k} = \gamma y_j^p \delta_k^p, \qquad \delta_k^p = F'(s_k^p)\sum_{o=1}^{N_o} \delta_o^p w_{k,o}$$

- At initial step $\delta_o^p = (o_o^p - y_o^p)F'(s_o^p)$
- Back propagation is **feasible** because the computation of deltas within one hidden layer **needs** only one **value of input from given neuron, weights as the input into following layer** and **values of deltas from previous recursive step**

Setting of Hyper-parameters

- Different results for various setting of algorithm
- Hyper-parameters
  - Validation - training ratio
  - Momentum
  - Batch size
  - Number of epochs
  - Learning rate
    - Fixed learning rate for each epochs
    - Adaptive learning rate method (ADAM, RMSPROP)
- Shortcoming of algorithm
  - Using different random seed you might obtain different results
    - Random initial weights
    - Randomness within algorithm (e.g. batching)

Practical Application - Neural Networks

- Our objective
  - j-th **expected yearly payment** $\mathrm{E}Y_{o,t,c,j} = \mu_{t,c,j}$ of claim o will be modeled with respect to reporting delay t and length of claim settlement c, where $j = 1, 2, \ldots, c$
  - Main objective is the modeling of **development patterns** $\boldsymbol{\mu}_{t,c} = (\mu_{t,c,1}, \mu_{t,c,2}, \ldots, \mu_{t,c,c})^T$
- Various structures, activation functions, number of neurons and approaches were used
- One stage approach
- Two stage approach
  - One model for probability of non-zero claim
  - Another model just for positive claims

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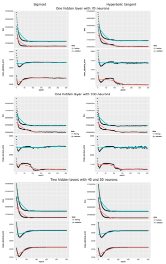
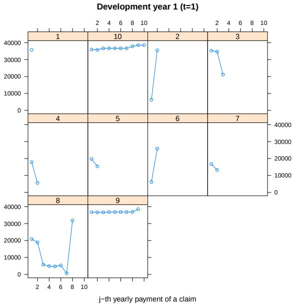
Practical Application - One Stage Approach

- Based on **validation loss function** (mean square error) model with model with **one hidden layer with 70 hyperbolic tangent neurons** output layer **ReLu** neuron

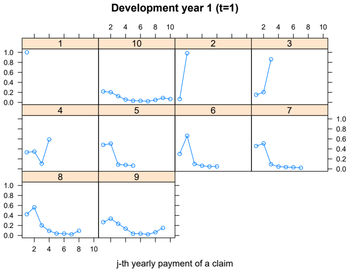


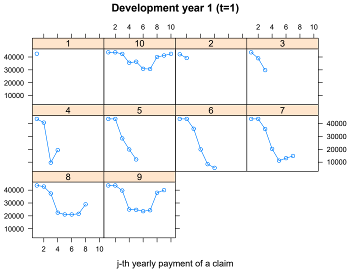


- Figure displays patterns $\mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \mu_{1,4}, \mu_{1,5}, \mu_{1,6}, \mu_{1,7}, \mu_{1,8}, \mu_{1,9}, \mu_{1,10}$
- Chosen approach is **not suitable**

Practical Application - Two Stage Approach

- Stage one - model with **one hidden layer of 20 sigmoid neurons** output neuron **sigmoid**
- Stage two - model with **one hidden layer of 90 sigmoid** output neuron **linear**

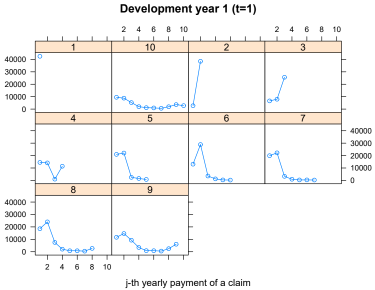


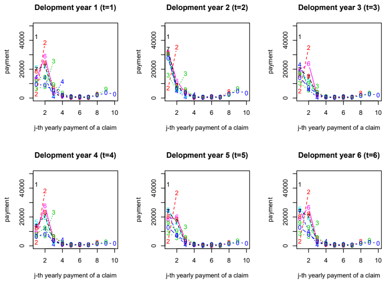


- **Reasonable** results

Practical Application - Two Stage Approach

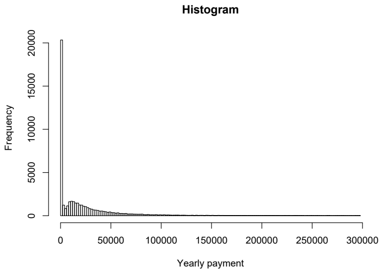
- Joined patterns for all development years/reporting delay





Hurdle Models

- Our data has mass at zero - reason for two stage approach within neural networks



- Hurdle model is proposed for such a situation

$$f_{hurdle}\left(Y_i = y_i\right) = \begin{cases} f_1\left(0\right), & y_i = 0 \\ \left(1 - f_1\left(0\right)\right)\frac{f_2(y_i)}{1-f_2(0)} = \Phi f_2\left(y_i\right) & y_i = 1, 2, \dots. \end{cases}$$

Poisson Hurdle Models

- Basic example of Poisson hurdle model

$$f_{hurdle}\left(Y_i = y_i\right) = \begin{cases} 1 - p_i, & y_i = 0 \\ p_i \frac{e^{\{-\lambda_i\}}\lambda_i^{y_i}}{\left(1 - e^{\{-\lambda_i\}}\right) y_i!} = \Phi f_2\left(y_i\right) & y_i = 1, 2, \ldots, \end{cases}$$

where p_i is modeled by **logistic regression**

$$\begin{aligned} L(\beta, \gamma) &= \left[\prod_{y_i=0} P(y_i = 0)\right]\left[\prod_{y_i>0} P(y_i > 0) f_2(y_i)\right] \\ &= \left[\prod_{y_i=0} \frac{1}{1 + e^{\mathbf{X}_i\beta}}\right]\left[\prod_{y_i>0} \frac{e^{\mathbf{X}_i\beta}}{1 + e^{\mathbf{X}_i\beta}} \frac{e^{-e^{\mathbf{Z}_i\gamma}} e^{\mathbf{Z}_i\gamma y_i}}{\left(1 - e^{-e^{\mathbf{Z}_i\gamma}}\right) y_i!}\right] \\ &= \left[\prod_{y_i>0} e^{\mathbf{X}_i\beta} \prod_{i=1}^{n} \frac{1}{1 + e^{\mathbf{X}_i\beta}}\right]\left[\prod_{y_i>0} \frac{e^{-e^{\mathbf{Z}_i\gamma}} e^{\mathbf{Z}_i\gamma y_i}}{\left(1 - e^{-e^{\mathbf{Z}_i\gamma}}\right) y_i!}.\right] \end{aligned}$$

- Due to the **independence** of the Y_i's, it is **possible to factor the likelihood function** into $L_1(\beta)$ and $L_2(\gamma)$
- The factorization allows us to **maximize** $L_1(\beta)$ and $L_2(\gamma)$ **separately**

Practical Application - Hurdle Models

- Due to the specifics of non-zero part of our data it can be **viewed as count data**
  - Poisson vs. negative binomial
  - Possible improvement Gamma ...
- Due to the **convergence** issue **merging last two levels** within covariates
- Based on AIC **negative binomial hurdle model** was chosen
- **Simplification of linear predictor** for hurdle component
  - Excluding development year/reporting delay (not statistically significant)

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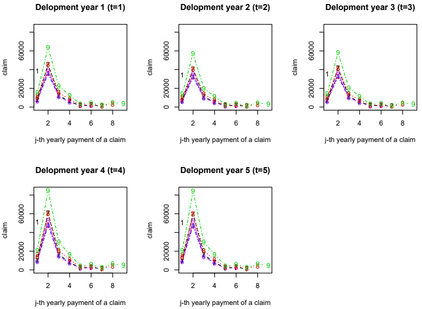
Practical Application - Hurdle Models

- Less flexible than patterns from neural networks due to the lower number of parameters - may lead to better prediction (further investigation and back-testing)
- Possible improvement by including interactions - leads to more flexible patterns



The End

Questions?

Thank you for your attention.

Michal Gerthofer

Further reading I

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