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Practical example of an Economic Scenario Generator

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Agenda

- Introduction

- Deterministic vs. stochastic approach

- Mathematical model

- Application of scenarios

- Calibration of input parameters

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Introduction

Introduction

There are a couple of purposes for which insurance companies use cash-flow models

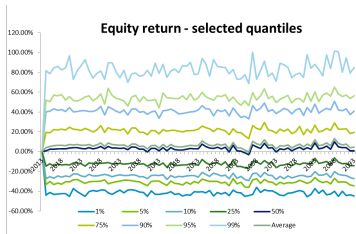- Solvency II
- MCEV
- LAT
- Planning (forecasts) etc.

For each of these the cash-flow model needs to be fed with assumptions on future macroeconomic development.

Deterministic models use one single (best-estimate) economic scenario which simulates the certainty equivalent (CEV) of future macroeconomic development.

Stochastic models capture the future uncertainty (volatility) by using a set of scenarios which simulate the assumed range of future macroeconomic development.

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Introduction

The idea behind using the economic scenarios is assuming that the future economic situation (esp. asset yields or inflation) can be described by a mathematical model with a given distribution and the set of scenarios is a random sample of this distribution.
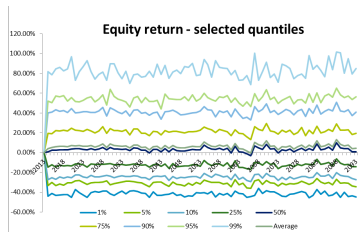

Equity return - selected quantiles

Consequently, using a stochastic model and economic scenarios can help to answer e.g. the following crucial questions:

- what is the value of options and guarantees embedded in my insurance portfolio?
- what is the probability that my company will be running at a loss in 20 years?
- what is the confidence interval for a participant's fund value after 60 years?
- etc.

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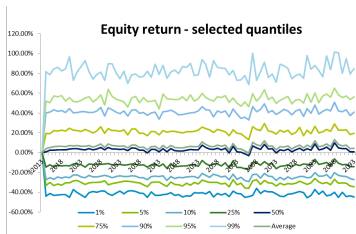
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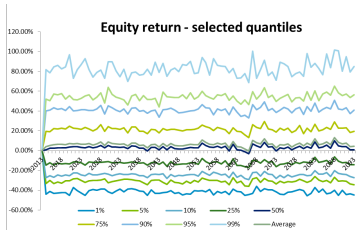

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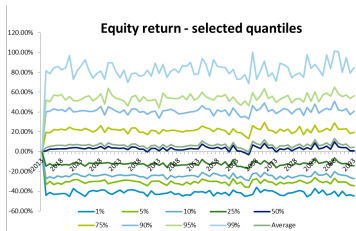

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Deterministic vs. stochastic approach

Overview

What are the general characteristics of an arbitrage-free [1] deterministic (CEV) economic scenario?

- all assets (bonds, equity) earn the same
- the time structure of the future yield curves is given by the initial yield curve (forward-spot relation)
- the discount rate is equal to the risk-free rate (1-yr forward)

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[1] *"No free lunch with vanishing risk"*

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Question:

Which of these are valid also for stochastic scenarios?

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Risk-neutral vs. real-world

First, it is necessary to introduce a further split of the stochastic approach.
Generally, there can be two types of stochastic scenarios:

- real-world
- risk-neutral

The real-world scenarios are the "natural" ones. As you would expect,
different assets have different volatility, and with higher volatility come
higher yields (risk premium). Moreover, the real-world scenarios reflect the
real probability distribution of economic variables.

However, it is not that straightforward with risk-neutral scenarios.

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Risk-neutral scenarios

The "most common" definition of risk neutrality is that all assets are expected to earn the risk free rate (i.e there is no risk premium even for volatile assets). Equivalently, if we denote by $Q(i,t)$ the value of a risky asset total return index in time t and by $Q(0,t)$ we denote the risk-free total return index in time t, we get:

$$\mathbb{E}\left[\frac{Q(i,t)}{Q(0,t)}\right] = 1 \qquad \forall i \tag{1}$$

This has to hold for risk-neutral scenarios. However, does that mean that the expected value of a risky asset total return index is equal to the expected value of the cash total return index? In other words, does the following also have to hold?

$$\mathbb{E}\left[Q(i,t)\right] = \mathbb{E}\left[Q(0,t)\right] \qquad \forall i \tag{2}$$

Risk-neutral scenarios

The answer is: not necessarily. If the last equation would hold, it would imply that:

$$Cov\left(Q(0,t), \frac{Q(i,t)}{Q(0,t)}\right) = 0 \qquad \forall i \tag{3}$$

However, as we could easily observe on the market, there is no reason why the return of a risky asset could not be correlated with the cash return.

Therefore, we have to be extra careful how to define the risk-neutrality, what does the definition imply and what can we expect from risk-neutral scenarios defined in such a way.

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Mathematical model

General

There are a number of mathematical models which aim to describe the behaviour of various macroeconomic indicators such as interest rates, inflation etc.

These models are usually based on an assumption that the behaviour of the selected indicators can be described by two components: a deterministic (non-random) function and a random noise (Wiener process) with zero mean.

Such a model then describes the development of the selected indicator using a ordinary (stochastic) differential equation (stochastic ODE):

$$\mathrm{d}r_t = \beta\left(t, r_t\right)\mathrm{d}t + \gamma\left(t, r_t\right)\mathrm{d}W_t$$

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Linear ODE's

A special case of a stochastic ODE's are linear stochastic ODE's in which the functions β and γ are linear functions, i.e. in a form of

$$
\begin{aligned}
\beta\left(t, r_t\right) &= a_t + b_t r_t \\
\gamma\left(t, r_t\right) &= c_t + \sigma_t r_t
\end{aligned}
$$

Such ODE's are special in a sense that there exists an analytical solution to these equations, i.e. the resulting process r_t can be explicitly calculated and does not need to be approached numerically.

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One-factor models

By selecting different coefficients *a*, *b*, *c* and σ one comes to various models which are often used, such as:

Gaussian random walk

$$\mathrm{d}r_t = \alpha r_t \mathrm{d}t + \sigma r_t \mathrm{d}W_t$$

Hull-White model of short-term interest rates

$$\mathrm{d}r_t = (\mu_t - \alpha r_t)\,\mathrm{d}t + \sigma r_t \mathrm{d}W_t$$

Naturally, there exist more comprehensive models which, apart from the deterministic and random components, take into account a third component: a jump process (usually Poisson or compound Poisson process). This process causes random jumps in the modelled indicator and thus approaches the sudden drops/rises which can be observed in reality.

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More advanced models

Moreover, there also exist so-called multi-factor short-rate models which do not take into account jumps but assume more than one source of randomness instead. An example of such models can be e.g.

Chen model of short term interest rates

$$
\begin{aligned}
\mathrm{d}r_t &= (\mu_t - \alpha_t)\,\mathrm{d}t + \sigma_t\,\sqrt{r_t}\mathrm{d}W_t \\
\mathrm{d}\alpha_t &= (\rho_t - \alpha_t)\,\mathrm{d}t + \sigma_t\,\sqrt{\alpha_t}\mathrm{d}W_t \\
\mathrm{d}\sigma_t &= (\omega_t - \sigma_t)\,\mathrm{d}t + \varphi_t\,\sqrt{\sigma_t}\mathrm{d}W_t
\end{aligned}
$$

Both the models with jump processes and the multi-factor models have one common characteristic: a very difficult and comprehensive calibration of input parameters. You not only have to have a reliable and sufficiently "long" data, but you also have to make additional assumptions on e.g. correlations.

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Selected model

The ESG which will be described is built on the Hull-White short-term interest rate model . The pros and cons of using such model:

Pros

- Compared to more comprehensive models it is more intuitive; the resulting process can be derived analytically
- The calibration of input parameters is reasonably straightforward
- It is capable of modelling the term structure of interest rates
- It is a model which is widely used in industry, it has been widely scrutinised

Cons

- The assumptions (volatility, correlations) are fixed (i.e. not time-dependent)
- Less extreme behaviour than observed on the market (in the short-term perspective)

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Model settings

The general approach to the modelling of individual assets / indicators will be:

- the risk-free return index (denote $Q(0, t)$) will be modelled by the Hull-White model
- the "infinitely" long bond return index (denote $Q(1, t)$), equity return indices (denote $Q(i, t), i = 2, \ldots, n$) and the real return index (denote $Q(n + 1, t)$) will be modelled by two components:
  - the risk-free return index
  - the risk premium above risk-free return, modelled by Gaussian random walk
- the ZCB price in time t with maturity period u (denote $P(t, u), t \le u$) will be modelled by combining the risk-free return index and the long bond return index

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Model settings

The previous description can be summarised by the following equation:

$$\frac{Q(i,t)}{Q(0,t)} = \frac{Q(i,0)}{Q(0,0)} \exp\left\{ \left(\mu_i - \frac{1}{2}V_{ii}\right)t + \sum_{j=1}^{i} \Lambda_{ij} W(j,t) \right\} \tag{4}$$

where:

- μ_i is an estimate of the risk premium of the given asset
- V_{ii} is an estimate of the given asset's variance on top of the risk-free asset variance
- $\Lambda = [\Lambda_{ij}]_{i,j=1}^{n+1}$ is the matrix of correlated volatilities
- $W(j,t)$ is a n dimensional Wiener process

Moreover, we can assume that $Q(0,0) = Q(i,0) = 1$.

Remark: the long bond return is in fact the return of the "holding bonds with given duration" strategy.

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Piece of math

In order to derive the formulas for $Q(0, t)$ and $P(t, u)$, it is necessary to make the following vital assumption:

$$\frac{dP(t, u)}{P(t, u)} = e^{-\alpha(u-t)} \frac{dQ(0, t)}{Q(0, t)} + \left(1 - e^{-\alpha(u-t)}\right) \frac{dQ(1, t)}{Q(1, t)} \tag{5}$$

As a result, we can derive the formulas for $Q(0, t)$ and $P(t, u)$

$$Q(0, t) = \frac{1}{P(0, t)} \exp \left\{ -\mu_1 \left(t - \frac{1 - e^{-\alpha t}}{\alpha} \right) - \Lambda_{11} W(1, t) + V_{11} X_t \right.$$
$$\left. + V_{11} \left(\frac{t}{2} - \frac{1 - e^{-2\alpha t}}{4\alpha} \right) \right\}$$

where $P(0, t)$ is the initial yield curve.

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Piece of math

The derived formula for $P(t, u), t \leq u$:

$$P(t,u) = \frac{P(0,u)}{P(0,t)} \exp\left\{\Lambda_{11}\left(1 - e^{-\alpha(u-t)}\right)X_t + \Lambda_{11}\frac{(1 - e^{-\alpha t})(1 - e^{-\alpha(u-t)})}{\alpha}\right.$$
$$\left. -V_{11}\left(1 - \frac{(1 - e^{-2\alpha t})(1 - e^{-2\alpha(u-t)})}{4\alpha}\right)\right\}$$

In both equations we used a process X_t. By this we denote an Ornstein-Uhlenbeck process defined by:

$$X_t = \int_0^t e^{-\alpha(t-s)} \mathrm{d}W(1,s) \tag{6}$$

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Piece of math

Using the previously mentioned equation of the Hull-White model (with constant parameters):

$$dr_t = (\mu - \alpha r_t)\, dt + \sigma r_t dW_t$$

and the Ito's lemma, we get

$$r_t = e^{-\alpha t} r(0) + \frac{\mu}{\alpha}\left(1 - e^{-\alpha t}\right) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$$

where the last term corresponds with our Ornstein-Uhlenbeck process X_t.

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Deflators

Generally, a deflator $D(t)$ is a stochastic process which satisfies the following equation

$$\mathbb{E}\left[D(t)Q(i,t)\right] = 1 \qquad \forall i$$

Naturally, there is a significant difference between deflators in case of risk-neutral and real-world scenarios. In our model, the deflator can be expressed in the following way:

$$D(t) = \frac{1}{Q(0,t)} \exp\left\{ -\sum_{j=1}^{n} \beta_j W(j,t) - \frac{1}{2} \sum_{j=1}^{n} \beta_j^2 t \right\}$$

where the parameters β_i satisfy the following:

$$\mu_i = \sum_{j=1}^{i} \Lambda_{ij} \beta_j$$

Deflators

If we recall the fact that for the risk-neutral scenarios the risk premiums μ_i are equal to 0, we obtain a simplified expression for risk-neutral scenarios.

$$D(t) = \frac{1}{Q(0, t)}$$

Here it is useful to note that the risk-neutral deflator is an analogy to the discount rate in the deterministic approach (i.e. both are constructed from the risk-free rate).

In case of real-world scenarios, one can easily see that the deflator can be split into two parts:

- "risk-neutral" part (represented by $Q(0, t)$)
- "compensation" part, combining the real-world probabilities and the risk-profile of $Q(i, t)$

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Application of scenarios

Overview

Generally, the use of scenarios was mentioned at the beginning of the presentation. However, now when we know more about the characteristics of risk-neutral and real-world scenarios, we can assess every situation from a better point of view.

Calculation of TVFOG

In case one needs to calculate the TVFOG embedded in an insurance (pension fund, ...) portfolio, both risk-neutral and real-world scenarios can be used. However, there are a few points in favour of risk-neutral scenarios:

- simplicity: the risk-neutral scenarios are simpler, with more transparent deflators
- lean approach: in case you don't care about the future cash-flows distribution and need only present values, it is easier to generate and use risk-neutral scenarios
- convergence: the convergence for risk-neutral scenarios (esp. for deflators) is much faster than for real-world scenarios

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Overview

Projection of future development

As already mentioned, there can be several purposes for which one needs to model the expected development of future cash flows, e.g.:

- estimation of a future event probability
- estimation of the confidence interval for a certain future value (fund, asset value, ...)
- etc.

For such projections, one needs the real probability distribution of economic variables. Hence, the real-world scenarios are the only correct option.

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Calibration of input parameters

Initial yield curve

As already mentioned, the model outputs (especially the shape of the yield curves and the equity return) significantly depend on the shape of the initial yield curve.

The initial yield curve has to be interpolated/extrapolated from the individual term's yields available. There are a couple of methods how to do it, and when selecting one of them, one has to be aware of what is the desired outcome.

I will briefly mention just two of the possible methods:

- Smith-Wilson method
- Nelson-Siegel-Svensson method

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Smith-Wilson method

One option how to interpolate/extrapolate the initial yield curve is the Smith-Wilson method.

This method is based on defining a set of special polynomials and calculating a linear combination of these so that the resulting (spot) yield curve "hits" the actual yields for individual terms.

Consequently, in case that the spot curve is not completely smooth, the implied forward curve may become "spiky".

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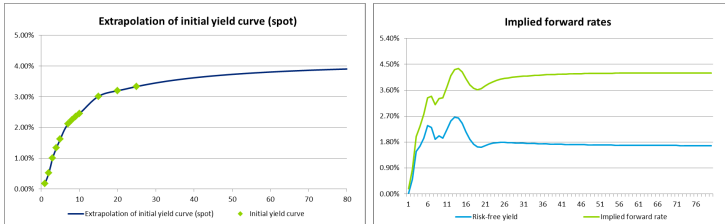
Nelson-Siegel-Svensson method

Another option how to interpolate/extrapolate the initial yield curve is the Nelson-Siegel-Svensson method.

This method is based on defining a set of smooth functions and estimating their parameters so that the mean squared distance between the initial yields and the modelled yields is minimised.

A big advantage of this method is that it produces smooth forward curve.

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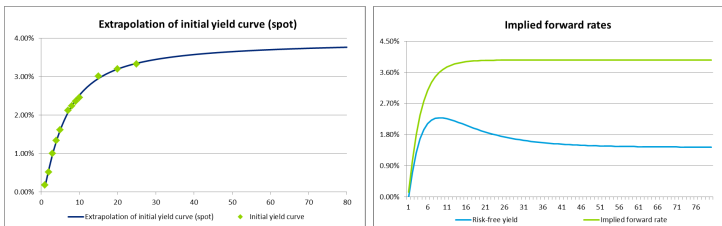
Input parameters

Regarding the input parameters for the ESG, there are basically two possible approaches:

- derivation of market-implied parameters
- derivation of parameters from historical data

Market-implied parameters

- these should be used if market values have to be replicated
- strongly depend on current market "mood"
- in some cases (e.g. inflation-linked bonds for deriving inflation parameters) may not be available

Parameters derived from historical data

- built on the assumption that the history will repeat itself
- does not take into account any expected development
- history may not be available (e.g. for emerging markets)

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Convergence

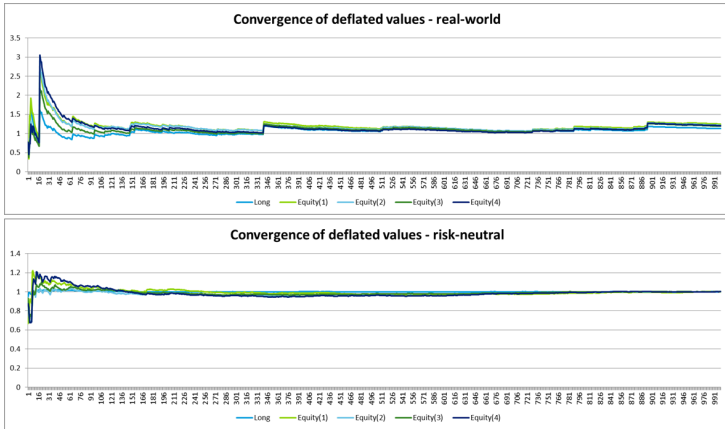As an example, the theoretical variance of the $Q(i, t)$ index has the following form:

$$Var[Q(i,t)] = \mathbb{E}[Q(i,t)]^2 \left(\exp\left\{ t(V_{ii} - \Lambda_{i1}\Lambda_{11} + V_{11}) + V_{11}\left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right) \right.\right.$$
$$\left.\left. + \left(\frac{1 - e^{-\alpha t}}{\alpha}\right)(2\Lambda_{i1}\Lambda_{11} - V_{11}) \right\} - 1 \right)$$

Consequently, the number of simulations required for convergence to the theoretical mean value (on $(1 - \alpha)\%$ confidence level) can be expressed as:

$$N = \frac{1}{\epsilon^2} \times \phi_{1-\alpha}^2 \times Var(Q(i,t)) \tag{7}$$

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Convergence

The following example demonstrates faster convergence of risk-neutral scenarios in case of deflated values of long bond return index and a couple of equity return indices.

Convergence

Generally, the model is sensitive to input data. Therefore, convergence may become a real problem if the input parameters are unadvisable. So, how you determine the "admissible" parameters?

In our particular model, the key indicator which determines the speed of convergence is the deflator, namely its "real-world part":

$$\exp\left\{-\sum_{j=1}^{n}\beta_j W(j,t) - \frac{1}{2}\sum_{j=1}^{n}\beta_j^2 t\right\}$$

However, what can be done when the parameters derived from hard data imply unfavourable risk parameters β_i?

There are basically two options: there do exist some variance reduction techniques, or one can consider adjusting the input parameters accordingto the Black-Litterman approach

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Convergence

Regarding variance reduction, the idea is that either the random numbers behind the scenario generation or the resulting scenarios are "adjusted" a bit in order to speed up the convergence.

- antithetic variates - using "opposite" paths to reduce variance
- quasi Monte Carlo - using low-discrepancy sequences
- scaling of scenarios - adjusting the resulting scenarios for the price of distorted characteristics

Alternatively, the Black-Litterman approach assumes that in an efficient market one might expect Sharpe ratios to be competed down. Consequently, risk premium is not required as an input and is calculated to satisfy some requirements on market equilibrium.

All the described approaches can be used in specific cases and definitely are not to be used automatically.

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