Calibrations of risk-neutral scenarios of interest rates in the Czech crown

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Abstract

The purpose of this paper is to show one possible approach how to construct a proxy for a term structure of volatilities in currencies (in particular in the Czech crown), where there is a limited interest rate derivatives market (swaption market, cap/floor market, etc.). The usual way how to get a particular term structure of volatility is from the market prices of interest rate derivatives by inverse engineering, i.e. extracting volatilities by equating the market price with the particular Black-Scholes formula for the derivative. This is a very common procedure in countries where such interest rate markets exist. However, in many countries (including the Czech Republic) there are no such markets and yet there is a great need (by banks, investment companies, insurance companies) to have such interest rate derivatives priced in its own currency.

The approach derived below describes the decomposition of the known (swaption or caplet) volatility into a semi-historical part and an implied part. While the semi-historical part can be computed for any unknown (swaption or caplet) volatility from a term stucture of interest rate, the implied part cannot be determined without a particular market. Therefore, there is no possible way to arrive at the precise implied volatility for currencies where there are no such markets. The question remains whether we can at least estimate or approximate these volatilities. In this paper we call the resulting quantities pseudo-implied volatilities.

1 Introduction

Since it is essential to have some idea of relationships between different market models we start with common assumptions concerning the price dynamics of zero-coupon bond

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and work our way up to the more complex market models such as LIBOR Market Model (LMM) and Swap Market Model (SMM). Along the way we come across interesting formulas and relationships some of which are important in our construction of term structure of volatilities.

In the following section on fundamentals we consider a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, P)$ with filtration $\mathbb{F} = \{\mathcal{F}(t); t \geq 0\}$ generated by a k-dimensional Brownian motion $W(t)_k = (W_1(t), W_2(t), \ldots, W_k(t))^T$.

2 Fundamentals

2.1 Bond-price dynamics

As we mentioned earlier we start with the usual assumption that the price of zero-coupon bond under the physical probability measure P is lognormally distributed. This means that the price at time t of T_j-zero-coupon bond (means zero-coupon bond maturing at time T_j) is driven by stochastic differential equation (SDE)

$$\mathrm{d}B(t, T_j) = \mu(t, T_j)B(t, T_j)\mathrm{d}t + B(t, T_j)\sigma^T(t, T_j)_k \mathrm{d}W(t)_k, \tag{1}$$

for maturities indexed by $j = 1, 2, \ldots, N$,where $0 \leq t < T_1 < T_2 < \ldots < T_N, \mu(t, T_j)$ is a non-random function of time t and $T_j, \sigma(t, T_j)_k = (\sigma_1(t, T_j), \sigma_2(t, T_j), \ldots, \sigma_k(t, T_j))^T$ is a non-random k- dimensional volatility term of $B(t, T_j)$ and finally $W(t)_k = (W_1(t), W_2(t), \ldots, W_k(t))^T$ is a k- dimensional Brownian motion under the physical probability measure P.

Although the non-random drift and diffusion coefficients can be estimated from the time series of zero-coupon bonds, it is not guaranteed that such estimated equation (1) will exclude arbitrage. It is well known that in the absence of the arbitrage the instantaneous rate of return of an asset must equal to the risk-free rate, in our case it is the short-term interest rate $\{r(t); t \geq 0\}$ used in the money market.

Therefore we wish to rewrite the previous equation (1) in the following way

$$\mathrm{d}B(t, T_j) = r(t)B(t, T_j)\mathrm{d}t + B(t, T_j)[(\mu(t, T_j) - r(t))\mathrm{d}t + \sigma^T(t, T_j)_k \mathrm{d}W(t)_k] \tag{2}$$

$$= r(t)B(t, T_j)\mathrm{d}t + B(t, T_j)\sum_{i=1}^{k} \sigma_i(t, T_j)(\Theta_i(t) + \mathrm{d}W_i(t)), \tag{3}$$

where the last equality (3) holds only if

$$\mu(t, T_j) - r(t) = \sum_{i=1}^{k} \sigma_i(t, T_j)\Theta_i(t), \tag{4}$$

or alternatively, only if

$$\mu(t,T_j) - r(t) = \sigma^T(t,T_j)_k \Theta(t)_k, \tag{5}$$

for all maturity dates $T_j \leq T_N$.

The components of $\Theta(t)_k$ are called market prices of risk for zero-coupon bond $B(t,T_j)$, for $T_j \leq T_N$. Notice that the number of independent Brownian motions dictates the number of market prices of risk.

According to the multidimensional Girsanov theorem (found for example in [5], page 224) we can define new probability measure

$$\widetilde{P}(A) = \int_A \widetilde{Z}(T)\mathrm{d}P(\omega), \tag{6}$$

for all $A \in \mathcal{F}(T)$, where the stochastic process $\{\widetilde{Z}(t); t \geq 0\}$ is the Radon-Nikodým derivative process

$$\begin{aligned} \widetilde{Z}(t) &= \frac{\mathrm{d}\widetilde{P}(\omega)}{\mathrm{d}P(\omega)}|_{\mathcal{F}(t)} \\ &= \exp\left\{ -\int_0^t \Theta^T(s)_k \mathrm{d}W(s)_k - \frac{1}{2}\int_0^t ||\Theta(s)_k||^2 \mathrm{d}s \right\}. \end{aligned}$$

Under this new probability measure $\widetilde{P}$, the k- dimensional stochastic process $\{\widetilde{W}(t)_k; t \geq 0\}$ defined

$$\widetilde{W}(t)_k = W(t)_k + \int_0^t \Theta(s)_k \mathrm{d}s,$$

or alternatively

$$\widetilde{W}_i(t) = W_i(t) + \int_0^t \Theta_i(s)\mathrm{d}s, \quad \text{for} \quad i = 1, 2, \ldots, k$$

is Brownian motion. Therefore the last equality (3) can be rewritten as

$$\mathrm{d}B(t,T_j) = r(t)B(t,T_j)\mathrm{d}t + B(t,T_j)\sigma^T(t,T_j)_k \mathrm{d}\widetilde{W}(t)_k, \tag{7}$$

for $T_j \leq T_N$.

2.2 LIBOR Market Model (LMM)

Since the forward LIBOR $\{L(t;T_j,T_{j+1}); t \geq 0\}$ is the simple interest rate defined

$$L(t;T_j,T_{j+1}) = \frac{1}{\Delta T_j}\left(\frac{B(t,T_j)}{B(t,T_{j+1})} - 1\right), \tag{8}$$

(see for example [5] , [2] or [4]), where $\Delta T_j = T_{j+1} - T_j$, one can derive the stochastic differential for (8) rather easily by applying the Itô-Doeblin formula, invoking new probability measure $\widetilde{P}^{T_{j+1}}$ (for full derivation see Appendix A.1) and using the equation in (7). We obtain

$$dL(t;T_j,T_{j+1}) = L(t;T_j,T_{j+1})\frac{1+\Delta T_j L(t;T_j,T_{j+1})}{\Delta T_j L(t;T_j,T_{j+1})}(\sigma^T(t,T_j)_k - \sigma^T(t,T_{j+1})_k)d\widetilde{W}^{T_{j+1}}(t)_k. \tag{9}$$

We can see that the forward LIBOR $\{L(t;T_j,T_{j+1}); t \leq T_j\}$ is $\widetilde{P}^{T_{j+1}}$-martingale with rather complicated volatility term, which we denote

$$\gamma(t,T_j)_k \stackrel{def}{=} \frac{1+\Delta T_j L(t;T_j,T_{j+1})}{\Delta T_j L(t;T_j,T_{j+1})}(\sigma(t,T_j)_k - \sigma(t,T_{j+1})_k) \tag{10}$$

This volatility term will be very important later in the paper. The desired LIBOR market model (LMM) is then the stochastic differential equation

$$dL(t;T_j,T_{j+1}) = L(t;T_j,T_{j+1})\gamma^T(t,T_j)_k d\widetilde{W}^{T_{j+1}}(t)_k \tag{11}$$

with volatility term $\gamma(t,T_j)_k$ prescribed directly. It is essential to understand that to get the LMM we keep the state variable $L(t;T_j,T_{j+1})$ but prescribe it the volatility $\gamma(t,T_j)_k$ directly (it is similar to prescribing the volatility to the forward rate in HJM model). In doing so, we get lognormally distributed forward LIBOR under the T_{j+1}-forward measure $\widetilde{P}^{T_{j+1}}$ with deterministic volatility vector $\gamma(t,T_j)_k$. Also since the forward LIBOR is lognormally distributed under $\widetilde{P}^{T_{j+1}}$, it makes it very easy to use such variable rate to price common interest rate derivatives such as swap, cap, floor, etc, see for example [5], [2], [1].

2.3 Interest rate swap & swap rate

Interest rate swap is a contract between two parties to exchange interest payments out of notional principal. Most interest rate swaps exchange floating-rate payments for fixed-rate payments and only the net payment is made. The party that pays the fixed-rate payment and receives the floating-rate payment holds payer's swap whereas the counter party holds receiver's swap.

Let us now price the payer swap. First, we denote by K the fixed rate, notional principal N, and $\langle T_m, T_n\rangle$ is the interval where all the payments are to be exchanged, i.e.

$$T_{m+1} < \ldots < T_{n-1} < T_n$$

are the payment dates.

Here we derive the price at time $t < T_m$ of the payer swap and denote it $SWAP_p(t;T_m,T_n,K,N)$. This will give us so called forward-starting payer swap. Notice that if $t = T_m$, then we get spot-starting payer swap. To price the payer swap we just need to price both incoming and outgoing cash flows. The outgoing fixed cash flows in the payer swap are depicted below in the Figure 1. Therefore the price at time $t < T_m$ of all fixed rate payments on $\langle T_m, T_n \rangle$ is

$$\begin{aligned} & -N \cdot K \cdot (\Delta T_m B(t, T_{m+1}) + \Delta T_{m+1} B(t, T_{m+2}) + \ldots + \Delta T_{n-1} B(t, T_n)) \\ & \qquad = -N \cdot K \sum_{j=m}^{n-1} \Delta T_j B(t, T_{j+1}). \end{aligned} \tag{12}$$

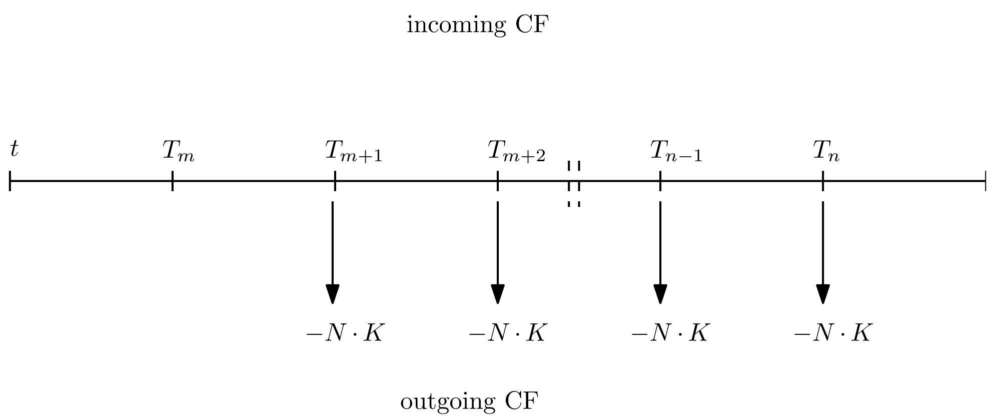

Figure 1: outgoing CF in the payer swap

As a floating rate we take the LIBOR. Because the forward LIBOR $L(t; T_j, T_{j+1})$ is the interest rate fixed at time t for borrowing or investing over the interval (T_j, T_{j+1}), the following series of „picture equations“ (Figure 2) hold for one floating-rate payment with $N = 1$ unit of currency (UoC).

Therefore, the series of the floating-rate payments can be rewritten as two fixed payments, see Figure 3, which value at time $t < T_m$ is

$$N(B(t, T_m) - B(t, T_n)). \tag{13}$$

Putting (12) and (13) together gives us desired pricing formula

$$SWAP_p(t; T_m, T_n, K, N) = N \left(B(t, T_m) - B(t, T_n) - K \sum_{j=m}^{n-1} \Delta T_j B(t, T_{j+1}) \right). \tag{14}$$

Of course, one would arrive at the same formula by just using the risk-neutral pricing formula and then changing to T_{j+1}-forward measure $\widetilde{P}^{T_{j+1}}$ and taking advantage of the fact that LIBOR is $\widetilde{P}^{T_{j+1}}$-martingale.

When the swap contract is initiated, its value is zero. This is logical, since no party is required to put up any upfront payments. The fixed interest rate K that make the swap

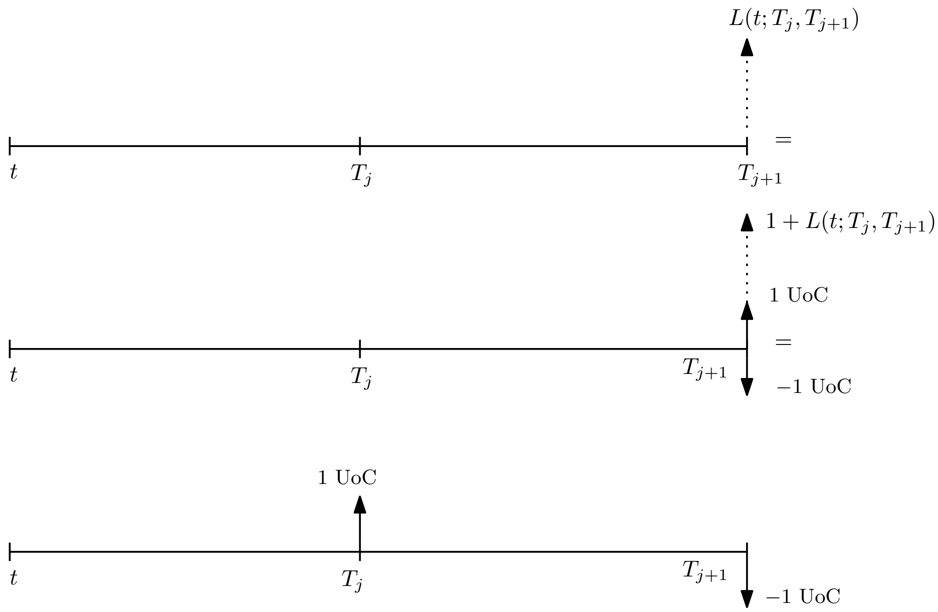

Figure 2: one floating-rate payment with $N = 1$

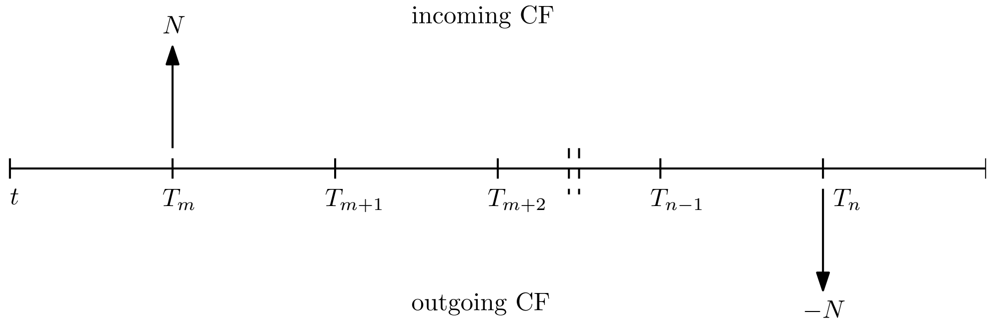

Figure 3: outgoing CF in the payer swap

have zero value is called swap rate and we denote it $R_{m,n}(t)$. Therefore the swap rate with tenor $\delta = T_n - T_m$ is from (14)

$$R_{m,n}(t) = \frac{B(t, T_m) - B(t, T_n)}{\sum_{j=m}^{n-1} \Delta T_j B(t, T_{j+1})}. \tag{15}$$

There is one important relationship that will be needed later on in this paper between the forward LIBOR and swap rate.

$$\begin{aligned} R_{m,n}(t) &= \frac{B(t,T_m) - B(t,T_n)}{\sum_{j=m}^{n-1} \Delta T_j B(t,T_{j+1})} \\ &= \frac{\sum_{j=m}^{n-1} (B(t,T_j) - B(t,T_{j+1}))}{\sum_{j=m}^{n-1} \Delta T_j B(t,T_{j+1})} \\ &= \frac{\sum_{j=m}^{n-1} L(t;T_j,T_{j+1}) \Delta T_j B(t,T_{j+1})}{\sum_{j=m}^{n-1} \Delta T_j B(t,T_{j+1})} \\ &= \sum_{j=m}^{n-1} \alpha_j(t) L(t,T_j,T_{j+1}), \end{aligned} \tag{16}$$

where in (16) we defined stochastic weight $\{\alpha_j(t); t \geq 0\}$ as

$$\alpha_j(t) = \frac{\Delta T_j B(t,T_{j+1})}{\sum_{j=m}^{n-1} \Delta T_j B(t,T_{j+1})} \quad , \, j = m, \, \ldots, \, n-1, \, n. \tag{17}$$

This relationship provides us a way to derive the dynamics of swap rate which in turn will be needed in pricing of swaptions.

2.4 Swap Market Model (SMM)

In this section, we use the last relationship (16) to derive the dynamics of swap rate $\{R_{m,n}(t); t \geq 0\}$ with tenor $\delta = T_n - T_m$. However, this relationship tells us that it is not possible for the swap rate to have lognormal distribution even under its own corresponding martingale measure. Nevertheless, we will show that by reasonable approximation one can get stochastic differential equation of the swap rate that is lognormally distributed and can be therefore used very well in pricing swaptions. Pricing errors that arise by such approximation are generally very small and can be in some instances smaller than the typical bid-ask spreads (see [4]). So we have

$$\begin{aligned} \mathrm{d}R_{m,n}(t) &= \mathrm{d}\left(\sum_{j=m}^{n-1} \alpha_j(t) L(t;T_j,T_{j+1})\right) \\ &= \sum_{j=m}^{n-1} (\alpha_j(t) \mathrm{d}L(t;T_j,T_{j+1}) + L(t;T_j,T_{j+1}) \mathrm{d}\alpha_j(t) + \mathrm{d}[\alpha_j, L](t)). \end{aligned} \tag{18}$$

After obtaining stochastic differential for $\{\alpha_j(t); t \geq 0\}$ in (17), differential of covariation of $\{\alpha_j(t); t \geq 0\}$ and $\{L(t;T_j,T_{j+1}), t \leq T_j\}$, defining new probability measure $\widetilde{P}^{SW}$ and some simple algebra (for full derivation see Appendix A.2), we get

$$
\mathrm{d}R_{m,n}(t) = R_{m,n}(t) \sum_{j=m}^{n-1} \frac{\alpha_j(t)L(t;T_j,T_{j+1})}{R_{m,n}(t)} \left(\sigma(t,T_{j+1})_k - \sigma_A(t)_k + \gamma(t,T_j)_k\right)^T \mathrm{d}\widetilde{W}^{SW}(t)_k, \tag{19}
$$

where we denoted by $\sigma_A(t)_k$ the volatility of the price at time $t < T_m$ of the unit cash-flow $\{A_{m,n}(t); t \geq 0\}$ occuring on the interval $\langle T_{m+1}, T_n \rangle$, i.e.

$$
A_{m,n}(t) = \sum_{j=m}^{n-1} \Delta T_j B(t, T_{j+1}),
$$

and

$$
\sigma_A(t)_k = \sum_{j=m}^{n-1} \alpha_j(t)\sigma(t, T_{j+1})_k. \tag{20}
$$

Altough $\{R_{m,n}(t); t \geq 0\}$ is $\widetilde{P}^{SW}$-martingale, it does not have any useful probability distribution, since its volatility term contains stochastic entities. To make $\{R_{m,n}(t); t \geq 0\}$ follow a lognormal distribution, we make following arguments. First of all it was shown in [4], that the stochastic coefficient

$$
\frac{\alpha_j(t)L(t;T_j,T_{j+1})}{R_{m,n}(t)}
$$

has a very low variability and therefore we can take it as a constant dependent on j, i.e. we put

$$
w_j = \frac{\alpha_j(t)L(t;T_j,T_{j+1})}{R_{m,n}(t)}, \quad \text{for} \quad j = m, \ldots, n-1.
$$

Furthermore, since $\sigma_A(t)_k$ is the stochastic weighted average (note, that α_j's are stochastic processes whereas $\sigma(t, T_{j+1})_k$ is deterministic) of deterministic $\sigma(t, T_{j+1})_k$, the contributions $\sigma(t, T_{j+1})_k - \sigma_A(t)_k$ to the whole sum in the volatility term of $\{R_{m,n}(t); t \geq 0\}$ are negligible. This could be seen from historically observed volatilities of zero-coupon bonds, see Table 1 for the relative errors (for $R_{1,m}(t)$) of these contributions. Therefore we arrive at the following approximation of (19)

$$
\begin{aligned}
\mathrm{d}R_{m,n}(t) &\approx R_{m,n}(t) \sum_{j=m}^{n-1} w_j \gamma^T(t,T_j)_k \mathrm{d}\widetilde{W}^{SW}(t)_k \\
&= R_{m,n}(t) \left(\sum_{j=m}^{n-1} w_j \gamma^T(t,T_j)_k \right) \mathrm{d}\widetilde{W}^{SW}(t)_k \\
&= R_{m,n}(t) \gamma_{m,n}^T(t)_k \mathrm{d}\widetilde{W}^{SW}(t)_k.
\end{aligned} \tag{21}
$$

Table 1: Relative errors

n	2	3	4	5	6	7	8
$R_{1,n}(12.08.2014)$	0,00359	0,00279	0,00300	0,00457	0,00446	0,00432	0,00417
$R_{1,n}(12.08.2015)$	0,00246	0,00278	0,00299	0,00456	0,00444	0,00430	0,00415

This approximation (21) is the famous Swap Market Model (SMM). Moreover, one can see that under the forward swap-rate probability measure $\widetilde{P}^{SW}$ the swap rate process $\{R_{m,n}(t); t \geq 0\}$ is now lognormally distributed, since the volatility term

$$\gamma_{m,n}(t)_k = \sum_{j=m}^{n-1} w_j \gamma(t, T_j)_k, \tag{22}$$

is at most some deterministic function of time. This fact is essential in pricing swaptions, as we'll see in the next section.

2.5 European swaption

European style swaption gives its holder the right but not the obligation to enter the interest rate swap contract at expiration date with a specific fixed interest rate K.

As before, let $SWAP_p(t; T_m, T_n, K, 1)$ denote the price of the payer swap at time $t < T_m < T_n$ with tenor $\delta = T_n - T_m$, fixed rate K and notional $N = 1$. Then the payoff function of the payer swaption at time T_m (since T_{m+1} is the first payment date in the swap) is

$$\begin{aligned} V_p(T_m) &= \max\{SWAP_p(T_m; T_m, T_n, K, 1), 0\} \\ &= (SWAP_p(T_m; T_m, T_n, K, 1))^+ \\ &= SWAP_p^+(T_m; T_m, T_n, K, 1), \end{aligned} \tag{23}$$

where $\max\{x, 0\} = x^+$.

Since we know the pricing formula for the payer swap (see 14), we can rewrite the previous payoff function in the following way

$$\begin{aligned} V_p(T_m) &= \left(B(T_m, T_m) - B(T_m, T_n) - K \sum_{j=m}^{n-1} \Delta T_j B(T_m, T_{j+1}) \right)^+ \\ &= (1 - B(T_m, T_n) - K A_{m,n}(T_m))^+ \\ &= (R_{m,n}(T_m) A_{m,n}(T_m) - K A_{m,n}(T_m))^+ \\ &= A_{m,n}(T_m) \left(R_{m,n}(T_m) - K \right)^+, \end{aligned} \tag{24}$$

where in the last but one equality we have used the formula for swap rate (15). This form of payoff function is more adequate for establishing the pricing formula for swaptions, since we can use the Swap Market Model derived in the previous section.

So the price of the payer swaption at time $t < T_m$ with expiry at T_m, fixed interest rate K and notional $N = 1$ is given by the risk-neutral pricing formula

$$
\begin{aligned}
SWAPTION_p(t; T_m, K, 1) &= \frac{1}{D(t)} \widetilde{\mathbb{E}} \left[D(T_m) V_p(T_m) \, | \mathcal{F}(t) \right] \\
&= \frac{1}{D(t)} \widetilde{\mathbb{E}} \left[D(T_m) A_{m,n}(T_m) (R_{m,n}(T_m) - K)^+ \, | \mathcal{F}(t) \right] \\
&= \frac{D(0) A_{m,n}(0)}{D(t) A_{m,n}(t)} A_{m,n}(t) \widetilde{\mathbb{E}} \left[\frac{D(T_m) A_{m,n}(T_m)}{D(0) A_{m,n}(0)} (R_{m,n}(T_m) - K)^+ \middle| \mathcal{F}(t) \right] \\
&= A_{m,n}(t) \frac{1}{\widetilde{Z}^{SW}(t)} \widetilde{\mathbb{E}} \left[\widetilde{Z}^{SW}(T_m) (R_{m,n}(T_m) - K)^+ \middle| \mathcal{F}(t) \right] \\
&= A_{m,n}(t) \widetilde{\mathbb{E}}^{SW} \left[(R_{m,n}(T_m) - K)^+ \middle| \mathcal{F}(t) \right], \qquad (25)
\end{aligned}
$$

where in the last two equalities we have used the Radon-Nikodým derivative process $\{\widetilde{Z}^{SW}(t); t \geq 0\}$ in (64) to transfer from the risk-neutral probability measure $\widetilde{P}$ to the forward swap-rate probability measure $\widetilde{P}^{SW}$ under which the swap rate $\{R_{m,n}(t); t \geq 0\}$ is a martingale and has a lognormal distribution. From here on, we have

$$
\begin{aligned}
SWAPTION_p(t; T_m, K, 1) = {} & A_{m,n}(t) \widetilde{\mathbb{E}}^{SW} \left[R_{m,n}(T_m) \chi_{\{R_{m,n}(T_m) > K\}} \middle| \mathcal{F}(t) \right] \\
& - K A_{m,n}(t) \widetilde{P}^{SW} \left(\{\omega; R_{m,n}(T_m) > K\} \right). \qquad (26)
\end{aligned}
$$

After calculating the first and the second term (see Appendix A.3) in (26) we arrive at Black-Scholes formula for the payer swaption at time $t < T_m$ (forward-starting payer swaption) with expiration at time T_m, fixed rate K and notional $N = 1$

$$
SWAPTION_p(t; T_m, K, 1) = A_{m,n}(t) \left[R_{m,n}(t) \Phi \left(d_+(R_{m,n}(t), \tau) \right) - K \Phi \left(d_-(R_{m,n}(t), \tau) \right) \right], \qquad (27)
$$

where

$$
\begin{aligned}
\Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} z^2 \right\} \mathrm{d}z = \int_{-y}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} z^2 \right\} \mathrm{d}z \\
d_{\pm}(R_{m,n}(t), \tau) &= \frac{1}{\sigma_{m,n} \sqrt{\tau}} \left[\ln \frac{R_{m,n}(t)}{K} \pm \frac{1}{2} \sigma_{m,n}^2 \tau \right], \\
\sigma_{m,n} &= \sqrt{\frac{1}{\tau} \int_t^{T_m} ||\gamma_{m,n}(s)_k||^2 \mathrm{d}s}, \\
\tau &= T_m - t.
\end{aligned}
$$

The prices of swaption are quoted in terms of $\sigma_{m,n}$ quantities, which are so-called Black swaption volatilities, where m specifies the expiration of the option on a swap and $n - m$ is the length (or tenor) of the swap contract.

3 Construction of the term structure of volatility

In this section, we show one possible way how to construct the term structure of (caplet) volatilities in CZK. And from those, we wll go on to construct the swaption volatilities (obviously again in CZK).

We will start with linearly interpolated swap-rate curve data from January 3rd, 2011 to August 12th, 2015 (see Table 2). Each row in Table 2 corresponds to the current swap-rate curve with tenor ranging from 1 year to 30 years.

Table 2: Swap-rate curve data in CZK (absolute values)

Date	$1Y$	$2Y$	$3Y$	...	$28Y$	$29Y$	$30Y$
3.1.2011	0,01450	0,02060	0,02315	...	0,03449	0,03430	0,03410
4.1.2011	0,01490	0,02020	0,02250	...	0,03359	0,03325	0,03290
5.1.2011	0,01460	0,02030	0,02250	...	0,03350	0,03320	0,03290
6.1.2011	0,01460	0,02045	0,02280	...	0,03414	0,03391	0,03368
7.1.2011	0,01460	0,02060	0,02270	...	0,03382	0,03359	0,03335
10.1.2011	0,01490	0,02090	0,02320	...	0,03382	0,03357	0,03333
11.1.2011	0,01490	0,02125	0,02350	...	0,03415	0,03405	0,03395
12.1.2011	0,01540	0,02200	0,02430	...	0,03504	0,03490	0,03475
13.1.2011	0,01540	0,02210	0,02470	...	0,03507	0,03501	0,03495
...	...	...	...	...	...	...	...
5.8.2015	0,00255	0,00340	0,00428	...	0,01659	0,01677	0,01695
6.8.2015	0,00255	0,00343	0,00428	...	0,01658	0,01677	0,01695
7.8.2015	0,00235	0,00330	0,00415	...	0,01585	0,01585	0,01585
10.8.2015	0,00250	0,00335	0,00418	...	0,01590	0,01590	0,01590
11.8.2015	0,00245	0,00330	0,00408	...	0,01597	0,01606	0,01615
12.8.2015	0,00245	0,00330	0,00405	...	0,01583	0,01594	0,01605

It is important to link these values to the denotation that we have established earlier in the paper. So, for example the first value 0.00245 on the swap-rate curve for August 12th, 2015 corresponds to $R_{0,1}(0)$, 0.0033 corresponds to $R_{0,2}(0)$ and so on. One must not mix rates such as $R_{0,1}(t)$ and $R_{0,2}(t)$ up with for example rates such as $R_{1,2}(t)$ and $R_{2,4}(t)$.

For the reasons explained below, we will need to bootstrap zero-coupon bond prices from these swap rates. This should not be complicated since from (15) we have

$$R_{0,n}(0) = \frac{1 - B(0,T_n)}{\sum_{j=0}^{n-1} \Delta T_j B(0,T_{j+1})} \tag{28}$$

Pluggin $n = 1$ in (28) gives (since $\Delta T_j = 1$)

$$B(0,T_1) = \frac{1}{1 + R_{0,1}(0)}, \tag{29}$$

for $n = 2$ we get

$$B(0,T_2) = \frac{1 - R_{0,2}(0)B(0,T_1)}{1 + R_{0,2}(0)}, \tag{30}$$

for $n = 3$ we get

$$B(0,T_3) = \frac{1 - R_{0,3}(0)(B(0,T_1) + B(0,T_2))}{1 + R_{0,3}(0)}, \tag{31}$$

and so on. In general, we get the following bootstrap formula

$$B(0,T_n) = \begin{cases} \dfrac{1}{1 + \Delta T_0 R_{0,1}(0)}, & n = 1, \\ \dfrac{1 - R_{0,n}(0)\sum_{j=0}^{n-2} \Delta T_j B(0,T_{j+1})}{1 + \Delta T_{n-1} R_{0,n}(0)}, & n > 1. \end{cases} \tag{32}$$

Applying (32) to each row of the swap-rate curve data yields zero-coupon bond prices as shown in Table 3.

Table 3: Bootstrapped zero-coupon bond price data in CZK

Date	$1Y$	$2Y$	$3Y$	...	$28Y$	$29Y$	$30Y$
3.1.2011	0,98571	0,95992	0,93335	...	0,38664	0,37718	0,36819
4.1.2011	0,98532	0,96069	0,93517	...	0,39874	0,39190	0,38554
5.1.2011	0,98561	0,96049	0,93517	...	0,39765	0,39009	0,38300
6.1.2011	0,98561	0,96021	0,93433	...	0,38720	0,37850	0,37024
7.1.2011	0,98561	0,95992	0,93462	...	0,39152	0,38291	0,37474
10.1.2011	0,98532	0,95936	0,93323	...	0,39329	0,38475	0,37666
11.1.2011	0,98532	0,95869	0,93240	...	0,38742	0,37639	0,36581
12.1.2011	0,98483	0,95727	0,93020	...	0,37916	0,36888	0,35904
13.1.2011	0,98483	0,95708	0,92909	...	0,37793	0,36617	0,35486
...	...	...	...	...	...	...	...
5.8.2015	0,99746	0,99323	0,98725	...	0,62126	0,60697	0,59271
6.8.2015	0,99746	0,99317	0,98725	...	0,62119	0,60681	0,59245
7.8.2015	0,99766	0,99343	0,98764	...	0,63650	0,62657	0,61679
10.8.2015	0,99751	0,99333	0,98755	...	0,63536	0,62541	0,61562
11.8.2015	0,99756	0,99343	0,98785	...	0,63332	0,62128	0,60932
12.8.2015	0,99756	0,99343	0,98794	...	0,63577	0,62330	0,61090

3.1 Term structure of caplet volatilities

Recall the LIBOR Market Model in (11). To simplify matters, we will consider case $k = 1$ in the remaining text of this paper. From [1] we know the caplet volatilities are quantities, denoted $\bar{\gamma}(T_j)$, that when plugged into Black-Scholes formula give us the market price of the caplets. These quantities are defined as

$$\bar{\gamma}(T_j) = \sqrt{\frac{1}{T_j}\int_0^{T_j} \gamma^2(t, T_j)\mathrm{d}t}, \quad j = 1, 2, \ldots, n-1, \tag{33}$$

where $\gamma(t, T_j)$ is the volatility of the forward LIBOR rate $L(t; T_j, T_{j+1})$.

It is essential to realize that the forward LIBOR rate $\{L(t; T_j, T_{j+1}); 0 \leq t \leq T_j\}$ is the stochastic process on the interval $t \in \langle 0, T_j\rangle$ (or variable rate on this interval), while it is constant on the interval $(T_j, T_{j+1}\rangle$. Therefore the volatility $\gamma(t, T_j)$ is a non-zero number only for $t \in \langle 0, T_j\rangle$, otherwise it is zero. For this very reason it is not possible to assume that quantities $\gamma(t, T_j)$ can be simply computed or rather estimated by statistical inference (as the square root of sample variance for example) from observed market values of LIBOR (or in our case PRIBOR rates) rates[1].

Let us assume that the volatility $\gamma(t, T_j)$ in (33) is constant (but nonzero) on the interval $\langle 0, T_j\rangle$ and denote it just γ_j instead of $\gamma(t, T_j)$. Then, for the caplet volatility we get

$$\bar{\gamma}(T_j) = \gamma_j, \quad j = 1, 2, \ldots, n-1. \tag{34}$$

Now, the question remains how do we compute or estimate γ_j's when we do not have market prices of caplets to imply them from. We can make use of the formula for $\gamma(t, T_j)$ in (10) that we have arrived at while deriving the LIBOR Market Model. Using the definition for LIBOR rate in (8) we get the following

$$\gamma(t, T_j) = \frac{B(t, T_j)}{B(t, T_j) - B(t, T_{j+1})}(\sigma(t, T_j) - \sigma(t, T_{j+1})) \tag{35}$$

By employing the statistics we can now estimate γ_j in the following way

$$\gamma_j = \frac{m_j(0)}{m_j(0) - m_{j+1}(0)}(\hat{\sigma}(T_j) - \hat{\sigma}(T_{j+1})), \tag{36}$$

where

$$m_j(0) = \frac{1}{S}\sum_{i=1}^{S} B_i^M(0, T_j), \quad \hat{\sigma}(T_j) = \sqrt{\ln\left(\frac{v_j(1)}{m_j^2(0)}e^{-2a_j(1)} + 1\right)}. \tag{37}$$

In (37), S denotes the number of swap-rate curves, $B_i^M(0, T_j)$ is the market price of the T_j-zero-coupon bond bootstrapped from the ith swap-rate curve and

$$a_j(1) = \ln\frac{m_j(1)}{m_j(0)}, \quad v_j(1) = \frac{1}{S-1}\sum_{i=1}^{S}(B_i^M(1, T_j) - m_j(1))^2. \tag{38}$$

[1] Note that this could be plausible since the change of probability measure does not effect the volatility but only the drift.

Notice that in (38) we also need $B_i^M(1,T_j)$ zero-coupon bonds. Those can be obtained from the relation $B^M(1,T_j) = \frac{B^M(0,T_j)}{B^M(0,1)}$. The first estimate in (37) is obvious, whereas the second is more elaborate and therefore its derivation is moved to the Appendix A.4.

Since (as shown in [2]) the term structure of caplet volatilities at time 0 (meaning today) is a set of ordered pairs

$$\left\{ (T_1, \gamma(0,T_1)) \, , \, (T_2, \gamma(0,T_2)) \, , \, \ldots , \, (T_n, \, \gamma(0,T_n)) \right\},$$

we have the approximation

$$\left\{ (T_1, \gamma_1) \, , \, (T_2, \gamma_2) \, , \, \ldots , \, (T_n, \, \gamma_n) \right\}$$

shown in Table 4. The corresponding graph, see Figure 4, takes the usual shape as seen in [1]. Plugging these quantities into Black-Scholes formula for caplet gives us the estimated today's market price of interest rate caplet, and also interest-rate cap, see [1] or [2] for particular pricing formulas.

Table 4: Approximated term structure of caplet volatilities (August 12th, 2015)

Year	γ_j
1	0.625187
2	0.575476
3	0.480692
4	0.413583
5	0.366853
6	0.330823
7	0.294691
8	0.277683
9	0.259092
10	0.261573
15	0.218426
20	0.242764
25	0.207357

3.2 Rebonato's simple approximation formula

In the previous section, we have shown how to estimate the term structure of caplet volatilities. In this section, we show that we can go even further and use the previous term structure in so-called Rebonato's approximation formula for swaption volatilities (since the first sketchy derivation was done by Rebonato in [3]) to estimate the term structure of swaption volatilities.

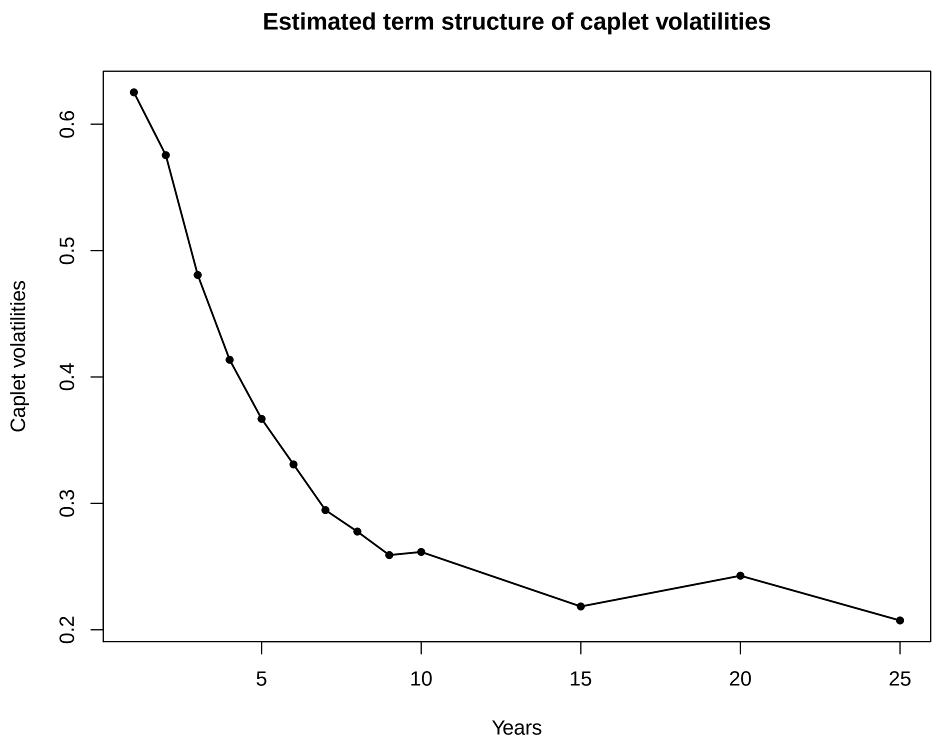

Figure 4: Graph of caplet volatilities (August 12th, 2015)

Recall the Swap Market Model (21), again with $k = 1$. In section 2.5, we have found that the square of the Black swaption volatility takes general form

$$\sigma_{m,n}^2 = \frac{1}{T_m - t} \int_t^{T_m} \gamma_{m,n}^2(s) \mathrm{d}s \tag{39}$$

for the forward-starting swaption. Since we want to compute the today's Black swaption volatilities, we simply substitute $t = 0$, which yields

$$\sigma_{m,n}^2 = \frac{1}{T_m} \int_0^{T_m} \gamma_{m,n}^2(s) \mathrm{d}s. \tag{40}$$

Because the stochastic differential equation for the logarithm of the swap rate is

$$\begin{aligned} \mathrm{d} \ln R_{m,n}(t) &= \frac{1}{R_{m,n}(t)} \mathrm{d}R_{m,n}(t) + \frac{1}{2} \left(-\frac{1}{R_{m,n}^2(t)} \right) \mathrm{d}[R_{m,n}, R_{m,n}](t) \\ &= \gamma_{m,n}(t) \mathrm{d}\widetilde{W}^{SW}(t) - \frac{1}{2} \gamma_{m,n}^2(t) \mathrm{d}t, \end{aligned}$$

we can immediately see that

$$\sigma_{m,n}^2 = \frac{1}{T_m} \int_0^{T_m} \gamma_{m,n}^2(s) \mathrm{d}s = \frac{1}{T_m} \int_0^{T_m} \mathrm{d}[\ln R_{m,n}, \ln R_{m,n}](s),$$

or more informally

$$\sigma_{m,n}^2 = \frac{1}{T_m} \int_0^{T_m} \mathrm{d}(\ln R_{m,n}(s)) \mathrm{d}(\ln R_{m,n}(s)). \tag{41}$$

Further, recall that the swap rate is essentially weighted average of the LIBOR rates, see (16)

$$R_{m,n}(t) = \sum_{j=m}^{n-1} \alpha_j(t) L_j(t),$$

where we use shorthand $L_j(t)$ for $L(t; T_j, T_{j+1})$ and

$$\alpha_j(t) = \frac{\Delta T_j B(t, T_{j+1})}{\sum_{j=m}^{n-1} \Delta T_j B(t, T_{j+1})}. \tag{42}$$

The first approximaton is following. We start by freezing all the weights α_j's at time $t = 0$

$$R_{m,n}(t) \approx \sum_{j=m}^{n-1} \alpha_j(0) L_j(t). \tag{43}$$

Brigo in [2] justifies this approximation by the fact that the variability of the α_j's is much smaller than the variability of the L_j's. This can be checked both historically and through simulation of the L_j's via Monte Carlo.

Taking the stochastic differential of the preceding approximation (43) gives

$$\begin{aligned} \mathrm{d}R_{m,n}(t) &\approx \mathrm{d}\left(\sum_{j=m}^{n-1}\alpha_j(0)L_j(t)\right) \\ &= \sum_{j=m}^{n-1}\alpha_j(0)\mathrm{d}L_j(t) \\ &= \sum_{j=m}^{n-1}\alpha_j(0)\gamma_j(t)L_j(t)\mathrm{d}\widetilde{W}^{T_{j+1}}(t), \end{aligned} \tag{44}$$

where in (44) we have used shorthand $\gamma_j(t)$ for $\gamma(t, T_j)$. Therefore,

$$(\mathrm{d}R_{m,n}(t))\,(\mathrm{d}R_{m,n}(t)) \approx \sum_{i=m}^{n-1}\sum_{j=m}^{n-1}\alpha_i(0)\alpha_j(0)\gamma_i(t)\gamma_j(t)L_i(t)L_j(t)\rho_{i,j}\mathrm{d}t,$$

where $\rho_{i,j}$ is the instantaneous correlation between the ith and jth LIBOR rate. Dividing this approximation by $R^2_{m,n}(t)$ yields

$$(\mathrm{d}\ln R_{m,n}(t))\,(\mathrm{d}\ln R_{m,n}(t)) \approx \sum_{i=m}^{n-1}\sum_{j=m}^{n-1}\frac{\alpha_i(0)\alpha_j(0)\gamma_i(t)\gamma_j(t)L_i(t)L_j(t)\rho_{i,j}}{R^2_{m,n}(t)}\mathrm{d}t.$$

Next approximation involves freezing all the L_j's (as was done earlier for the α_j's), which gives

$$(\mathrm{d}\ln R_{m,n}(t))\,(\mathrm{d}\ln R_{m,n}(t)) \approx \sum_{i=m}^{n-1}\sum_{j=m}^{n-1}\frac{\alpha_i(0)\alpha_j(0)L_i(0)L_j(0)\rho_{i,j}}{R^2_{m,n}(0)}\gamma_i(t)\gamma_j(t)\mathrm{d}t. \tag{45}$$

Putting this approximation back into formula (41) gives us the Rebonato's approximation formula for the Black swaption volatility

$$\sigma_{m,n} \approx \sigma^R_{m,n}, \tag{46}$$

where

$$\sigma^R_{m,n} = \sqrt{\frac{1}{T_m}\sum_{i=m}^{n-1}\sum_{j=m}^{n-1}\frac{\alpha_i(0)\alpha_j(0)L_i(0)L_j(0)\rho_{i,j}}{R^2_{m,n}(0)}\int_0^{T_m}\gamma_i(t)\gamma_j(t)\mathrm{d}t}. \tag{47}$$

3.3 Term structure of swaption volatilities

With the Rebonato's approximation formula for the Black swaption volatility in (47) we can approximate the term structure of swaption volatilities. However, before we make that step, we'll make further approximation by assuming constant γ_j's, which gives us

$$\sigma^R_{m,n} = \sqrt{\sum_{i=m}^{n-1}\sum_{j=m}^{n-1} \frac{\alpha_i(0)\alpha_j(0)L_i(0)L_j(0)\gamma_i\gamma_j\rho_{i,j}}{R^2_{m,n}(0)}}. \tag{48}$$

Since LIBOR (or in our case PRIBOR) rates are not available for tenors longer than one year, we can construct them using our bootstrapped zero-coupon bond price data in Table 3 using definition (8). The same goes for swap rates $R_{m,n}(0)$ (recall that the swap-rate curve data given in Table 2 are of the form $R_{0,n}(0)$) using definition (15).

The constructed LIBOR rates are in Table 8. Further according to (16) one has to construct a series of weights α_i's for each $R_{m,n}(0)$. One such table of weights is in Table 5. Since the γ_i's are collected from the term stucture of caplet volatilities in Table 4, the remaining quantities to be determined in (48) are instantaneous correlations $\rho_{i,j}$'s. Instantaneous correlations $\rho_{i,j}$'s can be historically estimated from the time series of the previously constructed LIBOR rates. Justification for this can be found in [2]. Excerpt from the historically estimated correlation matrix is in Table 6.

Table 5: Weights α_j's (August 12th, 2015)

	$\alpha_1(0)$	$\alpha_2(0)$	$\alpha_3(0)$	$\alpha_4(0)$	$\alpha_5(0)$	$\alpha_6(0)$	$\alpha_7(0)$	$\alpha_8(0)$
$R_{1,2}(0)$	1							
$R_{1,3}(0)$	0,50138	0,49862						
$R_{1,4}(0)$	0,33549	0,33364	0,33085					
$R_{1,5}(0)$	0,25274	0,25135	0,24925	0,24664				
$R_{1,6}(0)$	0,20325	0,20213	0,20044	0,19835	0,19581			
$R_{1,7}(0)$	0,17036	0,16941	0,16800	0,16624	0,16412	0,16184		
$R_{1,8}(0)$	0,14694	0,14613	0,14491	0,14339	0,14156	0,13960	0,13744	
$R_{1,9}(0)$	0,12945	0,12873	0,12766	0,12632	0,12470	0,12298	0,12107	0,11906

The resulting estimated Black swaption volatilities $\sigma^R_{m,n}$'s in CZK are in Table 7, where each row corresponds to the maturity of the option and each column to the length of the underlying swap.

3.4 Pseudo-implied Black swaption of volatilities

The estimated Black swaption volatilities in previous section 3.3 are partly obtained from historical data. Note that although we use current swap and LIBOR (PRIBOR in Czech Republic) rates in the estimation, the caplet volatilities entering the formula and the

Table 6: Excerpt from the historically estimated correlation matrix (in CZK, from January 3rd, 2011 to August 12th, 2015)

	$L_1(0)$	$L_2(0)$	$L_3(0)$	...	$L_{18}(0)$	$L_{19}(0)$	$L_{20}(0)$
$L_1(0)$	1,000000	0,973112	0,913039	...	0,527974	0,480488	0,431754
$L_2(0)$	0,973112	1,000000	0,974224	...	0,630941	0,584788	0,536882
$L_3(0)$	0,913039	0,974224	1,000000	...	0,722696	0,679252	0,633578
$L_4(0)$	0,859659	0,934789	0,983018	...	0,766321	0,724063	0,679368
...	...	...	...	...	...	...	...
$L_{17}(0)$	0,573491	0,674621	0,763205	...	0,997668	0,990775	0,979600
$L_{18}(0)$	0,527974	0,630941	0,722696	...	1,000000	0,997713	0,991031
$L_{19}(0)$	0,480488	0,584788	0,679252	...	0,997713	1,000000	0,997796
$L_{20}(0)$	0,431754	0,536882	0,633578	...	0,991031	0,997796	1,000000

Table 7: Term structure of swaption volatilities in CZK (August 12th, 2015)

	$1Y$	$2Y$	$3Y$	$4Y$	$5Y$	$6Y$	$7Y$	$8Y$	$9Y$
$1Y$	0,6252	0,6268	0,5349	0,4473	0,4390	0,4141	0,3862	0,3639	0,3427
$2Y$	0,5755	0,5153	0,4679	0,4305	0,4020	0,3752	0,3542	0,3355	0,3220
$3Y$	0,4807	0,4416	0,4094	0,3843	0,3561	0,3410	0,3239	0,3118	0,3022
$4Y$	0,4136	0,3868	0,3650	0,3429	0,3259	0,3105	0,3001	0,2917	0,2935
$5Y$	0,3669	0,3473	0,3267	0,3114	0,2975	0,2896	0,2816	0,2861	0,2661
$6Y$	0,3308	0,2932	0,2967	0,2843	0,2774	0,2718	0,2792	0,2582	0,2415
$7Y$	0,2947	0,2832	0,2724	0,2676	0,2633	0,2722	0,2480	0,2335	0,2296
$8Y$	0,2777	0,2648	0,2616	0,2581	0,2691	0,2415	0,2269	0,2236	0,2201
$9Y$	0,2591	0,2569	0,2538	0,2680	0,2354	0,2202	0,2149	0,2144	0,2110

Table 8: Constructed LIBOR rates

Forward rates	$L(0;0,1)$	$L(0;1,2)$	$L(0;2,3)$	...	$L(0;17,18)$	$L(0;18,19)$	$L(0;19,20)$
Denotation	$L_0(0)$	$L_1(0)$	$L_2(0)$	...	$L_{17}(0)$	$L_{18}(0)$	$L_{19}(0)$
3.1.2011	0,01450	0,02686	0,02846	...	0,04305	0,04393	0,04484
4.1.2011	0,01490	0,02563	0,02728	...	0,04434	0,04545	0,04661
5.1.2011	0,01460	0,02614	0,02707	...	0,03881	0,03929	0,03979
6.1.2011	0,01460	0,02645	0,02769	...	0,03370	0,03358	0,03347
7.1.2011	0,01460	0,02676	0,02707	...	0,03450	0,03450	0,03450
10.1.2011	0,01490	0,02706	0,02799	...	0,03897	0,03945	0,03994
11.1.2011	0,01490	0,02777	0,02819	...	0,03757	0,03790	0,03823
12.1.2011	0,01540	0,02879	0,02910	...	0,04105	0,04168	0,04233
13.1.2011	0,01540	0,02899	0,03013	...	0,03966	0,04015	0,04065
...	...	...	...	...	...	...	...
5.8.2015	0,00255	0,00425	0,00605	...	0,01868	0,01906	0,01944
6.8.2015	0,00255	0,00431	0,00599	...	0,01742	0,01766	0,01790
7.8.2015	0,00235	0,00425	0,00586	...	0,02039	0,02102	0,02167
10.8.2015	0,00250	0,00420	0,00585	...	0,01926	0,01977	0,02029
11.8.2015	0,00245	0,00415	0,00565	...	0,02142	0,02218	0,02297
12.8.2015	0,00245	0,00415	0,00556		0,02191	0,02276	0,02362

instantaneous correlations are obtained from historical discount curves and LIBOR rates respectively. Therefore one cannot expect that these estimated volatilities (both caplet volatilities and Black swaption volatilities in Tables 4 and 7 respectively) will match the implied market volatilities. Comparing estimated Black swaption volatilities (computed with our method) in EUR with market-implied Black swaption volatilities in EUR (collected from Bloomberg), see Tables 9 and 10, we can see that market-implied Black swaption volatilities in general tent to be higher.

Table 9: Estimated Black swaption volatilities in EUR (August 12th, 2015)

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	0,6284	0,4704	0,3935	0,3354
$5Y$	0,3312	0,3005	0,2927	0,2797
$10Y$	0,2733	0,2643	0,2620	0,2555

Table 10: Market-implied Black swaption volatilities in EUR (August 12th, 2015)

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	1,3212	0,7159	0,4826	0,3904
$5Y$	0,5809	0,4671	0,4024	0,3736
$10Y$	0,4763	0,4173	0,3865	0,3807

Table 11: Ratios $c_{m,n}$'s (August 12th, 2015)

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	2,1025	1,5219	1,2264	1,1640
$5Y$	1,7541	1,5545	1,3750	1,3357
$10Y$	1,7425	1,5790	1,4753	1,4902

If we denote by $c_{m,n}$ the ratio of the market-implied Black swaption volatility $\sigma_{m,n}$ and the corresponding estimated Black swaption volatility $\sigma^R_{m,n}$, i.e.

$$c_{m,n} = \frac{\sigma_{m,n}}{\sigma^R_{m,n}}, \tag{49}$$

we find out that $c_{m,n}$'s are for most combinations of m and n higher than one, see Table 11. Obviously, if the ratios $c_{m,n}$'s were equal exactly one then we would have $\sigma_{m,n} = \sigma^R_{m,n}$, which does not necessarily hold since the approximation in (46). In approximating $\sigma_{m,n}$'s by $\sigma^R_{m,n}$'s we have lost some information on the current market and haven't been able to obtain it back even by substituting historical estimations for the market implied entities.

Unfortunately, $c_{m,n}$'s are not available to us when there is no market for swaptions, but we can still compute $\sigma^R_{m,n}$'s, as it was shown in the last section. If we assume that the $c_{m,n}$'s computed from EUR quotes $\sigma_{m,n}$'s and $\sigma^R_{m,n}$'s are the same or at least very similar for any other European currency then by multiplying $\sigma^R_{m,n}$ computed for a particular currency by $c_{m,n}$ we obtain $\sigma^*_{m,n}$., i.e.

$$\sigma^*_{m,n} = \sigma^R_{m,n} \cdot c_{m,n}, \tag{50}$$

where we call $\sigma^*_{m,n}$ pseudo-implied Black swaption volatility[2]. Note that the product on the right-hand side of (50) holds for market-implied Black swaption volatilities if there is a market for swaptions. Also note that the left-hand side of (50) is in the correct currency, since $c_{m,n}$ on the right-hand side is a dimensionless number (meaning it has no units). For example pseudo-implied Black swaption volatilities in CZK are in Table 12.

Table 12: Pseudo-implied Black swaption volatilities in CZK (August 12th, 2015)

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	1,1247	0,6681	0,4736	0,3867
$5Y$	0,5730	0,4624	0,3872	0,3365
$10Y$	0,4725	0,3498	0,3156	0,3039

3.4.1 Time-inhomogeneity of $c_{m,n}$

It is natural to ask, whether we have to recalculate $c_{m,n}$ each time we want to calculate pseudo-implied Black swaption volatilities σ^*'s. One way to resolve the time dependency of $c_{m,n}$ is to look at the history of these values. If we take the Table 11 apart according to the option maturity and swap length and look at the time series of these components from June 6th, 2013 to August 8th, 2015, we obtain following three figures in Figure 5. All three figures indicates time dependence of $c_{m,n}$.

3.4.2 n-inhomogeneity of $c_{m,n}$

Looking at the Figure 5, one may notice the similar pattern of swap lengths across all swaption maturities. This feature can be taken advantage of, particularly in recent days, when $c_{m,n}$'s are very close to each other for given swaption maturities. Of course this is not always the case, as history indicates. But if we take simple average from $c_{m,n}$'s across all swap lengths for each swaption maturity, we obtain c_m's Table 13. Using this approach the resulting pseudo-implied Black swaption volatilities in CZK are then in Table 3.4.2. As mentioned, this approach is not always appropriate, since $c_{m,n}$ depends on n, but could be useful at times.

3.5 Pseudo-implied caplet volatilities

Here, we briefly discuss the computation of pseudo-implied caplet volatilities. The approach is much more simpler then with the pseudo-implied Black swaption volatilities, since

[2]The reason for this name is the same as for pseudo-random generators.

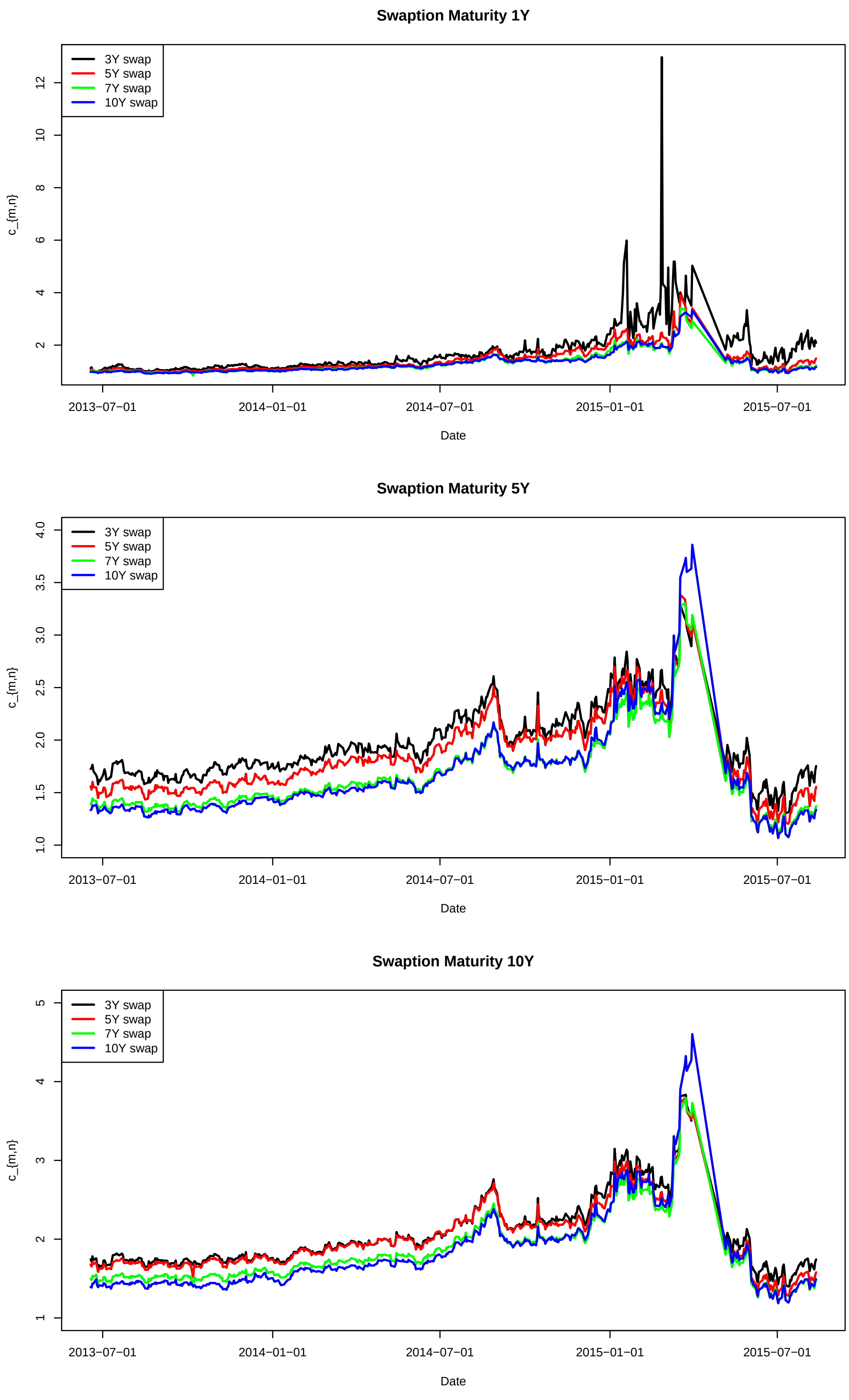

Figure 5: Ratios $c_{m,n}$ for swaption with various maturities and swap lengths

Table 13: Ratios c_m

m	c_m
$1Y$	1,5037
$5Y$	1,5048
$10Y$	1,5717

Table 14: Pseudo-implied Black swaption volatilities in CZK (August 12th, 2015)

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	0,8043	0,6601	0,5807	0,4996
$5Y$	0,4916	0,4476	0,4238	0,3791
$10Y$	0,4262	0,3482	0,3363	0,3206

we already have the semi-historical part from equation (36). Therefore applying the procedure described in the previous section (of course, instead of market implied swaption volatilities we use market implied caplet volatilities obtained from market implied cap volatilities by bootstrapping) gives us the desired pseudo-implied caplet volatilities.

4 Application to pricing swaptions

The following are numerical examples of swaption prices using either market-implied Black swaption volatilities (Table 10) or pseudo-implied Black swaption volatilities (Table 12). Note, that all the other arguments that are passed to the Black-Scholes formula (27) are in CZK (this means that $A_{m,n}$ is computed from Czech zero-coupon bond prices and we are using Czech swap-rate curve to compute $R_{m,n}$).In Table 15 we have ATM prices of a payer swaption with maturities 1Y, 3Y and 10Y, swap lengths 3Y, 5Y, 7Y and 10Y and nominal 1 CZK using market-implied Black swaption volatilities from Table 10, whereas

Table 15: ATM prices of a payer swaption (August 12th, 2015) based on Table 10

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	0,00877	0,01133	0,01304	0,01822
$5Y$	0,01949	0,02882	0,03750	0,05272
$10Y$	0,02035	0,04440	0,05637	0,07853

in Table 16 we have used pseudo-implied Black swaption volatilities from Table 12. The relative differences of the ATM prices of this payer swaption are shown in Table 17. They are computed as a difference between a particular element in Table 15 and the same element in Table 16 divided by the same element in Table 16.

One can see, that if for example an insurance company used our approach in pricing its products in CZK, in which embedded swaptions are present, it would save money in

Table 16: ATM prices of a payer swaption (August 12th, 2015) based on Table 12

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	0,00761	0,01060	0,01280	0,01805
$5Y$	0,01926	0,02855	0,03617	0,04773
$10Y$	0,02022	0,03798	0,04696	0,06402

Table 17: Relative differences (in %, August 12th, 2015)

	$3Y$	$5Y$	$7Y$	$10Y$
$1Y$	15,262%	6,864%	1,860%	0,948%
$5Y$	1,199%	0,927%	3,662%	10,446%
$10Y$	0,659%	16,881%	20,038%	22,650%

constructing its reserves, especially for products with longer maturity of the option and the swap length.

5 Conclusion

This paper has provided an approximation procedure for constructing a term structure of volatilities for countries where there is no interest-rate derivative market (we chose Czech Republic). This procedure is quite simple, in that it only uses the term structure of interest rates. Also note, that it uses a few approximations that might be far from permissible in relatively volatile markets. The resulting approximation formulas for each term structure are important to determine the approximated market price (in given currency) of many interest-rate derivatives.

6 Abbreviations and Notation

- LMM = LIBOR Market Model;
- HJM = Heath-Jarrow-Morton Model;
- SMM = Swap Market Model;
- SDE = Stochastic differential equation;
- UoC = Unit of currency;
- $B(t, T)$: Bond price at time t with maturity T;
- $r(t)$: Instantaneous spot interest rate at time t;
- P: Physical \Objective\Real-World probability measure;
- $W(t)_k$: k-dimensional Brownian motion under the Physical probability measure, i.e.
$$W(t)_k = (W_1(t),\, W_2(t),\, \ldots,\, W_k(t))^T;$$
- $\widetilde{P}$: Risk-neutral probability measure;
- $\widetilde{W}(t)_k$: k-dimensional Brownian motion under the Risk-neutral probability measure, i.e.
$$\widetilde{W}(t)_k = (\widetilde{W}_1(t),\, \widetilde{W}_2(t),\, \ldots,\, \widetilde{W}_k(t))^T;$$
- $L(t; S, T)$: forward LIBOR rate at time t for borrowing\investing over the interval (S, T);
- $\widetilde{P}^{T_{j+1}}$: T_{j+1}-forward probability measure, i.e.probability measure associated with the numeraire $B(\cdot, T_{j+1})$;
- $\widetilde{W}^{T_{j+1}}(t)_k$: k-dimensional Brownian motion under the T_{j+1}-forward probability measure, i.e.
$$\widetilde{W}^{T_{j+1}}(t)_k = (\widetilde{W}_1^{T_{j+1}}(t),\, \widetilde{W}_2^{T_{j+1}}(t),\, \ldots,\, \widetilde{W}_k^{T_{j+1}}(t))^T;$$
- $D(t)$: Stochastic discount factor at time t, i.e.
$$D(t) = \exp\left\{-\int_0^t r(s)\mathrm{d}s\right\};$$
- $R_{m,n}(t)$: Forward swap rate at time t for a swap with first reset date T_m and payment dates $T_{m+1},\, T_{m+2},\, \ldots,\, T_n$;

- $\widetilde{P}^{SW}$: forward swap-rate probability measure;
- $\widetilde{W}^{SW}(t)_k$: k-dimensional Brownian motion under the forward swap-rate probability measure, i.e.

$$\widetilde{W}^{SW}(t)_k = (\widetilde{W}_1^{SW}(t),\, \widetilde{W}_2^{SW}(t),\, \ldots,\, \widetilde{W}_k^{SW}(t))^T;$$

- $||\cdot||$: Euclidean norm, i.e. for example

$$||\sigma(t, T_{j+1})_k|| = \sqrt{\sum_{i=1}^{k} \sigma_i^2(t, T_{j+1})}$$

- $[X, X](t)$: quadratic variation of stochastic process $\{X(t); t \geq 0\}$;
- $[X, Y](t)$: cross variation of stochastic processes $\{X(t); t \geq 0\}$ and $\{Y(t); t \geq 0\}$;

A Appendix

A.1 LIBOR Market Model

Applying the Itô-Doeblin formula to the definition of the forward LIBOR in (8) gives us

$$\begin{aligned} \mathrm{d}L(t; T_j, T_{j+1}) &= \frac{1}{\Delta T_j}\mathrm{d}\left(\frac{B(t, T_j)}{B(t, T_{j+1})}\right) \\ &= \frac{1}{\Delta T_j}[B(t, T_j)\mathrm{d}B^{-1}(t, T_{j+1}) + B^{-1}(t, T_{j+1})\mathrm{d}B(t, T_j) + \mathrm{d}[B, B^{-1}](t)], \end{aligned} \tag{51}$$

where from (7) we have

$$\begin{aligned} \mathrm{d}B^{-1}(t, T_{j+1}) &= -\frac{1}{B^2(t, T_{j+1})}\mathrm{d}B(t, T_{j+1}) + \frac{1}{2}\left(\frac{2}{B^3(t, T_{j+1})}\right)\mathrm{d}[B, B](t) \\ &= B^{-1}(t, T_{j+1})(||\sigma(t, T_{j+1})_k||^2 - r(t))\mathrm{d}t - B^{-1}(t, T_{j+1})\sigma^T(t, T_{j+1})_k \mathrm{d}\widetilde{W}(t)_k, \end{aligned} \tag{52}$$

and

$$\mathrm{d}[B, B^{-1}](t) = -B(t, T_j)B^{-1}(t, T_{j+1})\sigma^T(t, T_j)_k \sigma(t, T_{j+1})_k \mathrm{d}t. \tag{53}$$

Putting (52) and (53) back into (51) gives us

$$\begin{aligned} \mathrm{d}L(t; T_j, T_{j+1}) =& \frac{1}{\Delta T_j}\frac{B(t, T_j)}{B(t, T_{j+1})}\left(||\sigma(t, T_{j+1})_k||^2 - \sigma^T(t, T_j)_k \sigma(t, T_{j+1})_k\right) dt \\ &+ \frac{1}{\Delta T_j}\frac{B(t, T_j)}{B(t, T_{j+1})}\left(\sigma^T(t, T_j)_k - \sigma^T(t, T_{j+1})_k\right)\mathrm{d}\widetilde{W}(t)_k. \end{aligned} \tag{54}$$

Now, since the discounted price of T_j-zero-coupon bond $\{D(t)B(t,T_j); t \leq Tj\}$ is $\widetilde{P}$-martingale, so is the stochastic process $\{\frac{D(t)B(t,T_j)}{D(t)B(t,T_{j+1})}; t \leq T_{j+1}\}$ but under different probability measure. We denote this new probability measure $\widetilde{P}^{T_{j+1}}$ and define it

$$\begin{aligned}\widetilde{P}^{T_{j+1}}(A) &= \int_A Z^{T_{j+1}}(T_{j+1})\mathrm{d}\widetilde{P}(\omega) \\ &= \int_A \frac{D(T_{j+1})B(T_{j+1},T_{j+1})}{D(0)B(0,T_{j+1})}\mathrm{d}\widetilde{P}(\omega) \\ &= \frac{1}{B(0,T_{j+1})}\int_A D(T_{j+1})\mathrm{d}\widetilde{P}(\omega),\end{aligned}$$

for all $A \in \mathcal{F}(T_{j+1})$.

One can see that by solving stochastic differetial equation for $\{D(t)B(t,T_{j+1}); t \leq T_{j+1}\}$ we will get the explicit form for the Radon-Nikodým derivative process $\{\widetilde{Z}^{T_{j+1}}(t); t \geq 0\}$

$$\begin{aligned}\widetilde{Z}^{T_{j+1}}(t) &= \frac{\mathrm{d}\widetilde{P}^{T_{j+1}}(\omega)}{\mathrm{d}\widetilde{P}(\omega)}|_{\mathcal{F}(t)} = \frac{D(t)B(t,T_{j+1})}{D(0)B(0,T_{j+1})} \\ &= \exp\left\{\int_0^t \sigma^T(s,T_{j+1})_k \mathrm{d}\widetilde{W}(s)_k - \frac{1}{2}\int_0^t ||\sigma(s,T_{j+1})_k||^2 \mathrm{d}s\right\}.\end{aligned}$$

Applying multidimensional Girsanov theorem gives us k-dimensional stochastic process $\{\widetilde{W}^{T_{j+1}}(t)_k; t \geq 0\}$ defined

$$\widetilde{W}^{T_{j+1}}(t)_k = \widetilde{W}(t)_k - \int_0^t \sigma(s,T_{j+1})_k \mathrm{d}s, \tag{55}$$

which is k-dimensional Brownian motion under the new probability measure $\widetilde{P}^{T_{j+1}}$. Using this new probability measure (sometimes called T_{j+1}-forward measure), we can rewrite the equation (54) as

$$\mathrm{d}L(t;T_j,T_{j+1}) = \frac{1}{\Delta T_j}\frac{B(t,T_j)}{B(t,T_{j+1})}(\sigma^T(t,T_j)_k - \sigma^T(t,T_{j+1})_k)\mathrm{d}\widetilde{W}^{T_{j+1}}(t)_k, \tag{56}$$

and finally using the definition of the forward LIBOR in (8) gives us

$$\mathrm{d}L(t;T_j,T_{j+1}) = L(t;T_j,T_{j+1})\frac{1+\Delta T_j L(t;T_j,T_{j+1})}{\Delta T_j L(t;T_j,T_{j+1})}(\sigma^T(t,T_j)_k - \sigma^T(t,T_{j+1})_k)\mathrm{d}\widetilde{W}^{T_{j+1}}(t)_k. \tag{57}$$

A.2 Swap Market Model

$$\begin{aligned} \mathrm{d}R_{m,n}(t) &= \mathrm{d}\left(\sum_{j=m}^{n-1}\alpha_j(t)L(t;T_j,T_{j+1})\right) \\ &= \sum_{j=m}^{n-1}\left(\alpha_j(t)\mathrm{d}L(t;T_j,T_{j+1}) + L(t;T_j,T_{j+1})\mathrm{d}\alpha_j(t) + \mathrm{d}[\alpha_j,L](t)\right), \end{aligned} \tag{58}$$

where for the stochastic weight $\{\alpha_j(t); t \geq 0\}$ we have (17), which we further simplify by writing

$$\alpha_j(t) = \frac{\Delta T_j B(t,T_{j+1})}{A_{m,n}(t)}, \tag{59}$$

where we put

$$A_{m,n}(t) = \sum_{j=m}^{n-1}\Delta T_j B(t,T_{j+1}), \tag{60}$$

which is essentially the price at time $t < T_m$ of the unit cash-flow occuring on the interval $\langle T_{m+1}, T_n\rangle$. The stochastic differential for $\{A_{m,n}(t); t \geq 0\}$ is then

$$\begin{aligned} \mathrm{d}A_{m,n}(t) &= \sum_{j=m}^{n-1}\Delta T_j \mathrm{d}B(t,T_{j+1}) \\ &= r(t)A_{m,n}(t)\mathrm{d}t + \sum_{j=m}^{n-1}\Delta T_j B(t,T_{j+1})\sigma^T(t,T_{j+1})_k \mathrm{d}\widetilde{W}(t)_k \\ &= r(t)A_{m,n}(t)\mathrm{d}t + A_{m,n}(t)\sum_{j=m}^{n-1}\frac{\Delta T_j B(t,T_{j+1})}{A_{m,n}(t)}\sigma^T(t,T_{j+1})_k \mathrm{d}\widetilde{W}(t)_k \\ &= r(t)A_{m,n}(t)\mathrm{d}t + A_{m,n}(t)\sum_{j=m}^{n-1}\alpha_j(t)\sigma^T(t,T_{j+1})_k \mathrm{d}\widetilde{W}(t)_k \\ &= r(t)A_{m,n}(t)\mathrm{d}t + A_{m,n}(t)\sigma_A^T(t)_k \mathrm{d}\widetilde{W}(t)_k, \end{aligned}$$

where we denoted

$$\sigma_A(t)_k = \sum_{j=m}^{n-1}\alpha_j(t)\sigma(t,T_{j+1})_k \tag{61}$$

as the vector of volatility terms of $\{A_{m,n}(t); t \geq 0\}$.

The stochastic differential for $\{A_{m,n}^{-1}(t); t \geq 0\}$ is then easily derived

$$\mathrm{d}A_{m,n}^{-1}(t) = A_{m,n}^{-1}(t)(||\sigma_A(t)_k||^2 - r(t))\mathrm{d}t + A_{m,n}^{-1}(t)\sigma_A^T(t)_k\mathrm{d}\widetilde{W}(t)_k, \tag{62}$$

and thus for $\{\alpha_j(t); t \geq 0\}$ we have

$$\begin{aligned}\mathrm{d}\alpha_j(t) &= \Delta T_j \mathrm{d}\left(B(t, T_{j+1})A_{m,n}^{-1}(t)\right)\\ &= \ldots\\ &= \alpha_j(t)\left(||\sigma_A(t)_k||^2 - \sigma_A^T(t)_k\sigma(t, T_{j+1})_k\right)\mathrm{d}t\\ &+ \alpha_j(t)\left(\sigma^T(t, T_{j+1})_k - \sigma_A^T(t)_k\right)\mathrm{d}\widetilde{W}(t)_k. \end{aligned}\tag{63}$$

At this point we can define a new probability measure, because we can take advantage of the fact that both $\{D(t)B(t, T_{j+1}); t \leq T_{j+1}\}$ and $\{D(t)A_{m,n}(t); t \geq 0\}$ are $\widetilde{P}$-martinagles. Thus also $\{\alpha_j(t); t \geq 0\}$ is a martingale but under different probability measure, we denote it $\widetilde{P}^{SW}$ and call it forward swap-rate probability measure. Its definition is

$$\begin{aligned}\widetilde{P}^{SW}(A) &= \int_A \widetilde{Z}^{SW}(T_m)\mathrm{d}\widetilde{P}(\omega)\\ &= \int_A \frac{D(T_m)A_{m,n}(T_m)}{D(0)A_{m,n}(0)}\mathrm{d}\widetilde{P}(\omega)\\ &= \frac{1}{A_{m,n}(0)}\int_A D(T_m)A_{m,n}(T_m)\mathrm{d}\widetilde{P}(\omega), \end{aligned}\tag{64}$$

for all $A \in \mathcal{F}(T_m)$.

The corresponding Radon-Nikodým derivative process $\{\widetilde{Z}^{SW}(t); t \geq 0\}$ can be derived as the solution to the stochastic differetial equation for $\{D(t)A_{m,n}(t); t \geq 0\}$. Applying multivariate Girsanov theorem gives us new k-dimensional stochastic process $\{\widetilde{W}^{SW}(t)_k; t \geq 0\}$ defined

$$\widetilde{W}^{SW}(t)_k = \widetilde{W}(t)_k - \int_0^t \sigma_A(s)_k\mathrm{d}s, \tag{65}$$

which is k-dimensional Brownian motion under the forward swap-rate probability measure $\widetilde{P}^{SW}$. Therefore the last equality (63) can be rewritten as

$$\mathrm{d}\alpha_j(t) = \alpha_j(t)\left(\sigma^T(t, T_{j+1})_k - \sigma_A^T(t)_k\right)\mathrm{d}\widetilde{W}^{SW}(t)_k. \tag{66}$$

Next, the differential of covariation of $\{L(t; T_j, T_{j+1}); t \leq T_j\}$ and $\{\alpha_j(t); t \geq 0\}$ is

$$\mathrm{d}[\alpha_j, L](t) = L(t; T_j, T_{j+1})\alpha_j(t)\gamma^T(t, T_j)_k\left(\sigma(t, T_{j+1})_k - \sigma_A(t)_k\right)\mathrm{d}t. \tag{67}$$

Putting (67), (66) and (11) under $\widetilde{P}$ into (58) gives us

$$
\begin{aligned}
\mathrm{d}R_{m,n}(t) =& -\sum_{j=m}^{n-1} \alpha_j(t)L(t;T_j,T_{j+1})\gamma^T(t,T_j)_k\sigma(t,T_{j+1})_k\mathrm{d}t \\
&+\sum_{j=m}^{n-1} \alpha_j(t)L(t;T_j,T_{j+1})\gamma^T(t,T_j)_k\mathrm{d}\widetilde{W}(t)_k \\
&+\sum_{j=m}^{n-1} \alpha_j(t)L(t,T_j,T_{j+1})\left(\sigma^T(t,T_{j+1})_k-\sigma_A^T(t)_k\right)\mathrm{d}\widetilde{W}^{SW}(t)_k \\
&+\sum_{j=m}^{n-1} \alpha_j(t)L(t;T_j,T_{j+1})\gamma^T(t,T_j)_k(\sigma(t,T_{j+1})_k-\sigma_A(t)_k)\mathrm{d}t \\
=&\sum_{j=m}^{n-1} \alpha_j(t)L(t;T_j,T_{j+1})\gamma^T(t,T_j)_k\left(-\sigma_A(t)_k\mathrm{d}t+\mathrm{d}\widetilde{W}(t)_k\right) \\
&+\sum_{j=m}^{n-1} \alpha_j(t)L(t;T_j,T_{j+1})\left(\sigma^T(t,T_{j+1})_k-\sigma_A^T(t)_k\right)\mathrm{d}\widetilde{W}^{SW}(t)_k \\
=&\sum_{j=m}^{n-1} \alpha_j(t)L(t;T_j,T_{j+1})\left(\sigma^T(t,T_{j+1})_k-\sigma_A^T(t)_k+\gamma^T(t,T_j)_k\right)\mathrm{d}\widetilde{W}^{SW}(t)_k \\
=&R_{m,n}(t)\sum_{j=m}^{n-1} \frac{\alpha_j(t)L(t;T_j,T_{j+1})}{R_{m,n}(t)}\left(\sigma^T(t,T_{j+1})_k-\sigma_A^T(t)_k+\gamma^T(t,T_j)_k\right)\mathrm{d}\widetilde{W}^{SW}(t)_k.
\end{aligned}
$$

A.3 European swaption

To calculate the first term in (26), we need to solve the Swap Market Model (21). Applying Itô-Doeblin formula to $\{\ln R_{m,n}(t); t \geq 0\}$ and integrating over the interval $\langle t, T_m\rangle$ gives us

$$
R_{m,n}(T_m) = R_{m,n}(t)\exp\left\{\int_t^{T_m} \gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k - \frac{1}{2}\int_t^{T_m} ||\gamma_{m,n}(s)_k||^2\mathrm{d}s\right\}. \qquad (68)
$$

Since $R_{m,n}(t)$ is $\mathcal{F}(t)$-measurable random variable and the exponential function on the right side of (68) does not depend on the σ-algebra $\mathcal{F}(t)$, we can rewrite the conditional expactation in (26)

$$
\widetilde{\mathbb{E}}^{SW}\left[R_{m,n}(t)\exp\left\{\int_t^{T_m} \gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k - \frac{1}{2}\int_t^{T_m} ||\gamma_{m,n}(s)_k||^2\mathrm{d}s\right\}\chi_{\{R_{m,n}(T_m)>K\}}\middle|\,\mathcal{F}(t)\right]
$$

as the unconditional one

$$
\widetilde{\mathbb{E}}^{SW}\left(x\exp\left\{\int_t^{T_m} \gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k - \frac{1}{2}\int_t^{T_m} ||\gamma_{m,n}(s)_k||^2\mathrm{d}s\right\}\chi_{\{R_{m,n}(T_m)>K\}}\right), \qquad (69)
$$

where we put x instead of $R_{m,n}(t)$.

Further, since

$$\widetilde{\mathbb{E}}^{SW}\left[\int_t^{T_m}\gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k\middle|\mathcal{F}(t)\right]=0$$

$$\widetilde{\mathbb{D}}^{SW}\left[\int_t^{T_m}\gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k\middle|\mathcal{F}(t)\right]=\int_t^{T_m}||\gamma_{m,n}(s)_k||^2\mathrm{d}s,$$

putting

$$\sigma_{m,n}=\sqrt{\frac{1}{(T_m-t)}\int_t^{T_m}||\gamma_{m,n}(s)_k||^2\mathrm{d}s} \tag{70}$$

we have the random variable

$$Z=\frac{\int_t^{T_m}\gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k}{\sigma_{m,n}\sqrt{T_m-t}}\sim N(0;1),$$

and therefore

$$\int_t^{T_m}\gamma_{m,n}^T(s)_k\mathrm{d}\widetilde{W}^{SW}(s)_k=Z\sigma_{m,n}\sqrt{T_m-t}.$$

The last unconditional expectation (69) can be thus further simplified

$$\begin{aligned}&\widetilde{\mathbb{E}}^{SW}\left(x\exp\left\{\sigma_{m,n}\sqrt{T_m-t}Z-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\chi_{\{R_{m,n}(T_m)>K\}}\right)\\&=\widetilde{\mathbb{E}}^{SW}\left(x\exp\left\{\sigma_{m,n}\sqrt{T_m-t}Z-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\chi_{\{Z>c\}}\right),\end{aligned} \tag{71}$$

where we put

$$c=\frac{1}{\sigma_{m,n}\sqrt{T_m-t}}\left[\ln\frac{K}{x}+\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right].$$

This last expectation (71) can be easily evaluated

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{SW}\left(x\exp\left\{\sigma_{m,n}\sqrt{T_m-t}Z-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\chi_{\{Z>c\}}\right)\\
&\quad=x\int_\Omega\exp\left\{\sigma_{m,n}\sqrt{T_m-t}Z(\omega)-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\chi_{\{\omega;Z(\omega)>c\}}(\omega)\mathrm{d}\widetilde{P}^{SW}(\omega)\\
&\quad=x\int_\Omega\exp\left\{\sigma_{m,n}\sqrt{T_m-t}Z(\omega)-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\chi_{(c,\infty)}(Z(\omega))\mathrm{d}\widetilde{P}^{SW}(\omega)\\
&\quad=x\int_{\mathbb{R}}\exp\left\{\sigma_{m,n}\sqrt{T_m-t}z-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\chi_{(c,\infty)}(z)\mathrm{d}\widetilde{P}_Z^{SW}(z)\\
&\quad=x\int_c^\infty\exp\left\{\sigma_{m,n}\sqrt{T_m-t}z-\frac{1}{2}\sigma_{m,n}^2(T_m-t)\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}\mathrm{d}z\\
&\quad=\dots\\
&\quad=x\cdot\Phi\left(-c+\sigma_{m,n}\sqrt{T_m-t}\right),
\end{aligned}\tag{72}$$

where

$$\Phi(y)=\int_{-\infty}^y\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}\mathrm{d}z=\int_{-y}^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}\mathrm{d}z$$

is the distribution function of the normal variable with mean zero and standard deviation one.

Putting back $R_{m,n}(t)$ in place of x, we get

$$\widetilde{\mathbb{E}}^{SW}\left[R_{m,n}(T_m)\chi_{\{R_{m,n}(T_m)>K\}}\middle|\mathcal{F}(t)\right]=R_{m,n}(t)\Phi\left(-c+\sigma_{m,n}\sqrt{T_m-t}\right).\tag{73}$$

The second term in (26) is quite easy to evaluate

$$\begin{aligned}
\widetilde{P}^{SW}\left(\{\omega;R_{m,n}(T_m)>K\}\right)&=\widetilde{P}^{SW}\left(\{\omega;Z(\omega)>c\}\right)\\
&=\int_{\{\omega;Z(\omega)>c\}}\mathrm{d}\widetilde{P}^{SW}(\omega)\\
&=\int_\Omega\chi_{\{\omega;Z(\omega)>c\}}(\omega)\mathrm{d}\widetilde{P}^{SW}(\omega)\\
&=\int_\Omega\chi_{(c,\infty)}(Z(\omega))\mathrm{d}\widetilde{P}^{SW}(\omega)\\
&=\int_{\mathbb{R}}\chi_{(c,\infty)}(z)\mathrm{d}\widetilde{P}_Z^{SW}(z)\\
&=\int_c^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}\mathrm{d}z\\
&=\Phi\left(-c\right).
\end{aligned}\tag{74}$$

Putting (72) and (74) back into (26) gives us

$$SWAPTION_p(t; T_m, K, 1) = A_{m,n}(t)R_{m,n}(t)\Phi\left(-c + \sigma_{m,n}\sqrt{T_m - t}\right) - KA_{m,n}(t)\Phi\left(-c\right). \tag{75}$$

Finally, let us put

$$\begin{aligned}
\tau &= T_m - t, \\
-c &= \frac{1}{\sigma_{m,n}\sqrt{T_m - t}}\left[\ln\frac{K}{R_{m,n}(t)} + \frac{1}{2}\sigma_{m,n}^2(T_m - t)\right] \\
&= \frac{1}{\sigma_{m,n}\sqrt{\tau}}\left[\ln\frac{R_{m,n}(t)}{K} - \frac{1}{2}\sigma_{m,n}^2\tau\right] \\
&= d_-(R_{m,n}(t), \tau), \\
-c + \sigma_{m,n}\sqrt{T_m - t} &= \frac{1}{\sigma_{m,n}\sqrt{\tau}}\left[\ln\frac{R_{m,n}(t)}{K} + \frac{1}{2}\sigma_{m,n}^2\tau\right] \\
&= d_+(R_{m,n}(t), \tau).
\end{aligned}$$

So the Black-Scholes formula for the payer swaption at time $t < T_m$ (forward-starting payer swaption) with expiration at time T_m, fixed rate K and notional $N = 1$ is

$$SWAPTION_p(t; T_m, K, 1) = A_{m,n}(t)R_{m,n}(t)\Phi\left(d_+(R_{m,n}(t), \tau)\right) - KA_{m,n}(t)\Phi\left(d_-(R_{m,n}(t), \tau)\right),$$

where

$$\begin{aligned}
d_\pm(R_{m,n}(t), \tau) &= \frac{1}{\sigma_{m,n}\sqrt{\tau}}\left[\ln\frac{R_{m,n}(t)}{K} \pm \frac{1}{2}\sigma_{m,n}^2\tau\right], \\
\sigma_{m,n} &= \sqrt{\frac{1}{\tau}\int_t^{T_m} ||\gamma_{m,n}(s)_k||^2 \mathrm{d}s}, \\
\tau &= T_m - t.
\end{aligned}$$

A.4 Term structure of caplet volatilities

Since γ_j in (36) does not depend on t, neither does σ in the same equation. Thus we can rewrite the stochastic differential for $B(t, T_j)$ in (1), after dropping the subscript k (since $k = 1$), in the following way

$$\mathrm{d}B(t, T_j) = \mu(t, T_j)B(t, T_j)\mathrm{d}t + B(t, T_j)\sigma(T_j)\mathrm{d}W(t). \tag{76}$$

Solving this stochastic differential equation gives

$$B(t, T_j) = B(0, T_j)e^{X(t,T_j)}, \tag{77}$$

where

$$X(t,T_j) = \int_0^t \mu(s,T_j)\mathrm{d}s - \frac{1}{2}\sigma^2(T_j)t + \sigma(T_j)\sqrt{t}Z,$$
$$= a_j(t) - \frac{1}{2}\sigma^2(T_j)t + \sigma(T_j)\sqrt{t}Z, \quad Z \sim N[0;1],$$

where we put $a_j(t) = \int_0^t \mu(s,T_j)\mathrm{d}s$. Therefore $X(t,T_j) \sim N[a_j(t) - \frac{1}{2}\sigma^2(T_j)t; \sigma^2(T_j)t]$ and we have

$$\mathbb{E}(B(t,T_j)) = B(0,T_j)e^{a_j(t)}, \tag{78}$$

$$\mathbb{D}(B(t,T_j)) = B^2(0,T_j)\left(e^{\sigma^2(T_j)t} - 1\right)e^{2a_j(t)}. \tag{79}$$

Using the moment method we get

$$m_j(t) = B(0,T_j)e^{a_j(t)}, \tag{80}$$

$$v_j(t) = B^2(0,T_j)\left(e^{\sigma^2(T_j)t} - 1\right)e^{2a_j(t)}, \tag{81}$$

where

$$m_j(t) = \frac{1}{S}\sum_{i=1}^{S} B_i(t,T_j),$$
$$v_j(t) = \frac{1}{S-1}\sum_{i=1}^{S} (B_i(t,T_j) - m_j(t))^2.$$

After some algebra, we obtain estimates

$$a_j(t) = \ln\frac{m_j(t)}{B(0,T_j)}, \tag{82}$$

$$\sigma(T_j) = \left[\frac{1}{t}\ln\left(\frac{v_j(t)}{B^2(0,T_j)}e^{-2a_j(t)} + 1\right)\right]^{\frac{1}{2}} \tag{83}$$

Although we are free to choose t and calculate the corresponding $B_i(t,T_j)$ from the relationship $B_i(t,T_j) = \frac{B_i(0,T_j)}{B_i(0,t)}$, it is safer to put $t = 1$, since it is the closest to the present time.

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