APRIL 2024

PUBLIC

SCENARIO GENERATOR

Practice & Examples

BUDOUCNOST JSTE VY

KB Pojišťovna

CONTENTS

1

INTRO

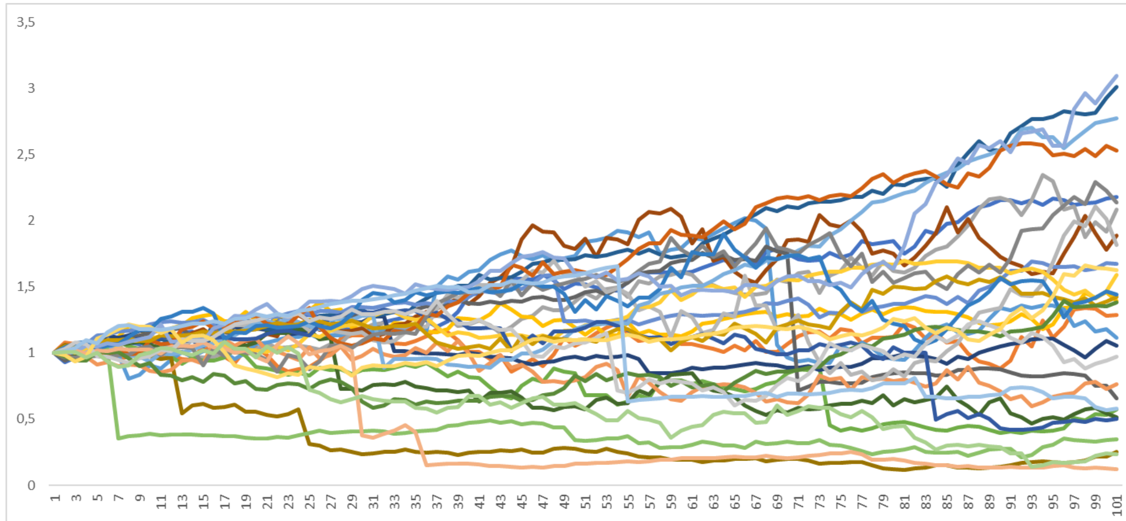
KB Pojišťovna

WHAT IS AN ECONOMIC SCENARIO GENERATOR (ESG)

ESG = tool for simulation of future evolution of macroeconomic and financial environment*

Typically, the simulation concerns:

- Real and nominal yield curves
- Equity indexes / investment fund returns
- Dividend return
- Credit spreads
- Inflation index (derived)



Each market variable modelled via specific random process.

Characteristics of many realizations of these processes are then investigated – usually thousands of scenarios.

*Although, Google's results for "ESG" relates mainly to sustainability ☺

OFF-TOPIC

Why did the economic scenario generator start focusing on ESG factors?

Because it realized it needed to incorporate a little "environmental humor" to balance out all those market fluctuations! ☺

RISK-NEUTRAL VS REAL-WORLD SCENARIOS

Two basic setups of ESG:

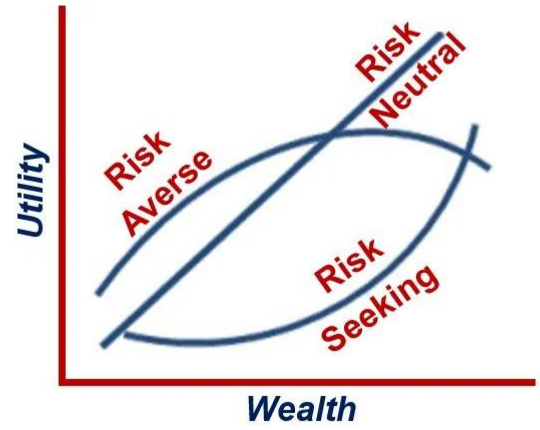
	Risk-neutral	Real-world
Objective / function	Market-consistent valuation	To generate realistic projection of assets and liabilities
Discount rate	All assets expected to earn risk-free rate	Riskier assets earn higher expected risk-adjusted return (risk-free rate plus risk premium)
Usage	Replicating market prices using simulation, valuation of options and derivatives, asset/liability pricing	ALM, risk management applications: assessing regulatory capital, “what-if” scenarios, value at risk

Real-world scenarios can be used for valuation (in theory) – difference in the used probability measure, usage of deflators (stochastic discount factors):

$$V = E^Q[d_T X] = E^P[D(T)X],$$

where V is the present value of future cashflow X at time T, Q means risk-neutral probability measure, P means real-world measure, $D()$ is the deflator and d is risk-free discount rate

RISK-NEUTRAL WORLD IN MORE DETAIL



Assumptions:

- Market participants do not require compensation for taking on risk.
- No arbitrage opportunities exist in the market

Key Concepts:

- Future cash flows discounted at the risk-free rate
- Probabilities of future events adjusted to reflect risk-neutral probabilities.
- Under the risk-neutral measure, the expected discounted value of an asset equals its current price.

Application:

- Commonly used in pricing derivatives such as options, where future payoffs are uncertain.
- Risk premium does not need to be considered in the valuation process.

Advantage:

- Simplifies calculations by assuming a single risk-free rate for discounting.

Conclusion: **We can assume risk-neutral world for pricing (valuation) of derivatives such as options with the valid result in real-world which is not risk-neutral.**

Image: adapted from math.stackexchange.com

APPLICATION IN INSURANCE COMPANIES

Valuation of options and guarantees in saving life insurance policies.

Option embedded in an insurance policy -> client behavior depends on certain (e.g. market) factors

$$BEL = PVFCF^{STOCH} = PVFCF^{CE} + TVFOG$$

BEL: Best estimate of liabilities

$PVFCF^{STOCH}$: Average of present values of future cash flows using stochastic scenarios

$PVFCF^{CE}$: Present value of future cash flows under certainty equivalent (deterministic) scenario

$TVFOG$: Time value of financial options and guarantees embedded in insurance contracts

Example of guarantees:

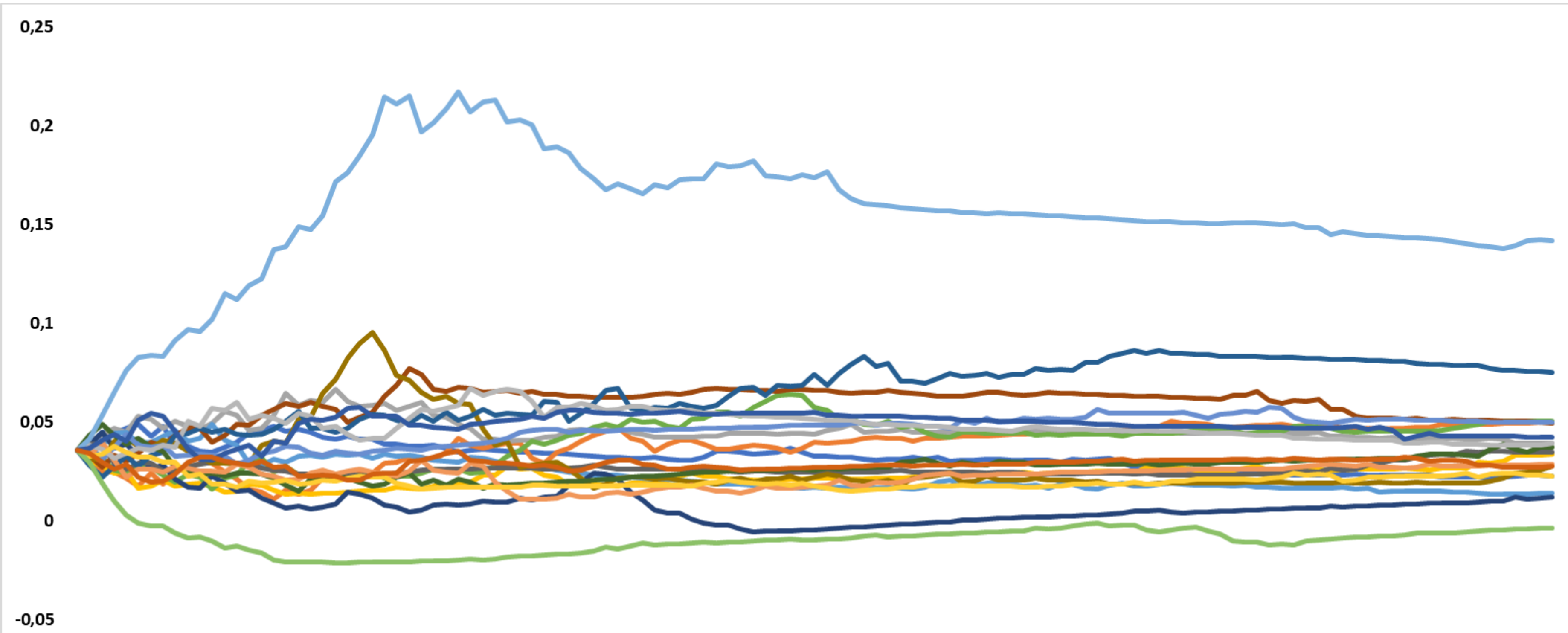
- Technical interest rate, guaranteed rate, minimum guaranteed death benefit

Examples of options:

- Profit share (to discretion of the company), surrender option (possibility to lapse), partial withdrawal

MONTE CARLO APPLICATION

- Monte Carlo simulations = sampling from distributions of uncertain parts of the model
- One scenario = one simulated market future = a path for the behavior of assets and liabilities
- Determination of TVFOG: by taking average of all discounted scenarios



-> time

2

MODEL FROM PRACTICE

KB Pojišťovna

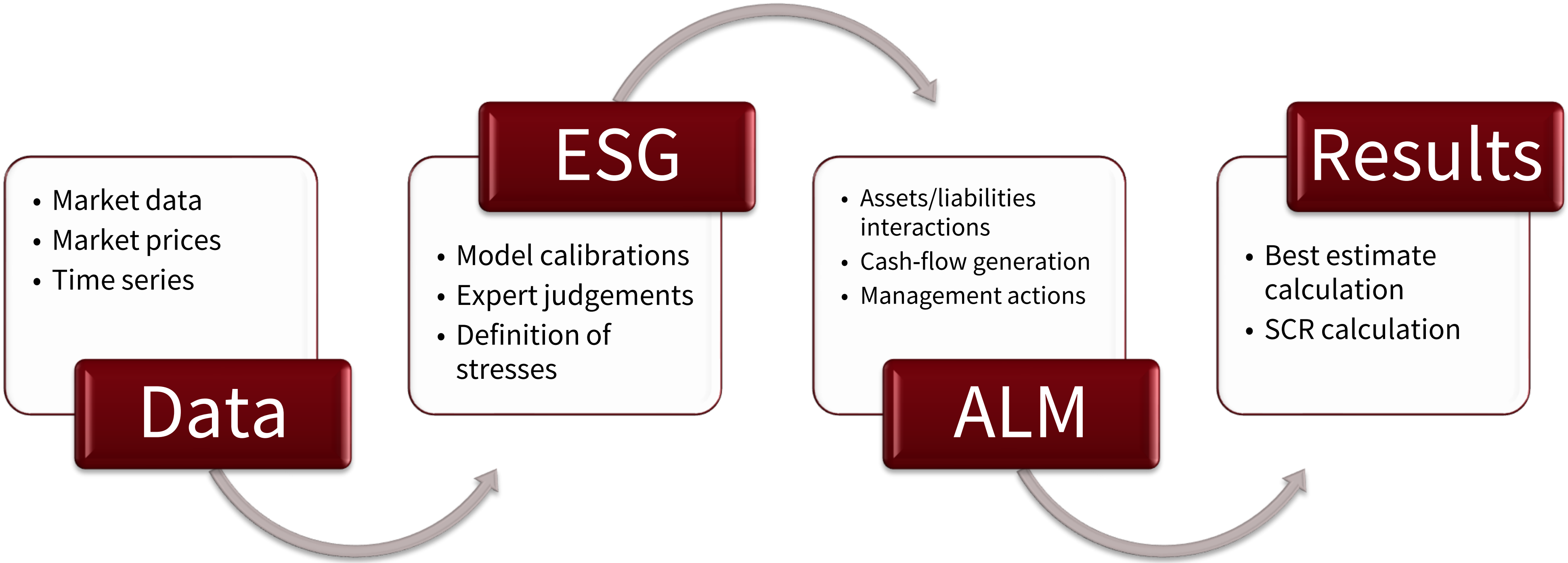
SCENARIO GENERATOR EXAMPLE

Set up of a scenario generator may look like this:

Economic variable	Model used
Nominal interest rates	Libor Market Model Plus (LMM+)
Real interested rates	Vašíček model with two factors
Equity index (e.g. Eurostoxx 50)	Stochastic Volatility Jump Diffusion (SVJD)
Real estate fund index	Fixed Volatility Model (Black-Scholes)
Dividend index	Vašíček (one factor)
Inflation index	Derived from nominal and real interest rate

Many other models exist for the individual economic variables (less or more complex).

SCENARIO GENERATOR EXAMPLE



ORDINARY (STOCHASTIC) DIFFERENTIAL EQUATION

Common form of ESG:

$$(next\ period\ value) = \ (current\ value) + (expected\ drift) + (random\ shock)$$

$Expected\ drift$ … calibrated to the central expectation of the change in value

$Random\ shock$ … calibrated to volatility in value

Model description for variable X in the form of stochastic differential equation:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t),$$

where:

$\mu(X(t), t)$ … drift coefficient

$\sigma(X(t), t)$ … volatility coefficient

$W(t)$ … Wiener process (independent and normally distributed increments)

INTEREST RATE MODELS

Basic types (with examples):

Short rate models

- One-factor models: Vašíček, CIR
- Multi-factor models: Hull-White

Forward rate models

- HJM model

Instantaneous rates not observable on the market (approximated by 2-week rate)

Market models

- LIBOR Market model (+ variations)
- Swap Market model

More complex models

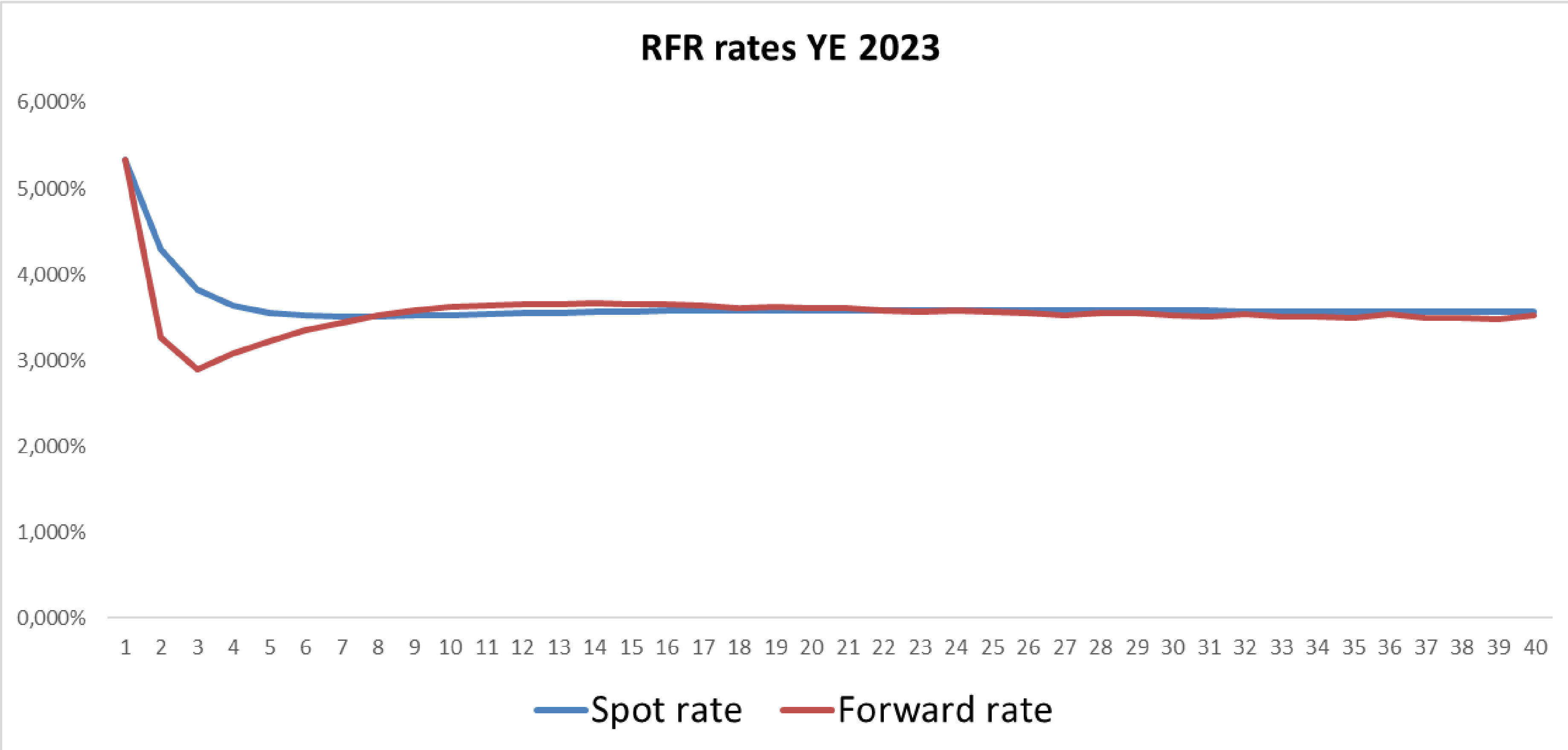
LIBOR MARKET MODEL (LMM) FOR YIELD CURVES

- Originally called „BGM“ model (names of authors)
- LIBOR in the title does not refer specifically to LIBOR, but it can indicate another interbank rate, for example EURIBOR or PRIBOR
- Belongs to so called „market models“ class of models*
- LMM model = set of stochastic differential equations for forward rates F_i quoted in the market
  - **Modelling of entire forward curve**

- Valuations based on LMM have to be done by means of Monte Carlo simulations

**https://www.columbia.edu/~mh2078/market_models.pdf*

LIBOR MARKET MODEL (LMM) FOR YIELD CURVES



LIBOR MARKET MODEL (LMM) FOR YIELD CURVES

Assume given tenor structure (equally spaced with intervals δ):

$$0 = T_0 < T_1 < \cdots < T_N$$

Let $F_i(t)$ be forward rate in the time t over period $[T_{i-1}, T_i]$.

For example, if we consider model for 40 years with monthly steps ⇒ $N = 40 \times 12 = 480$

Relationship between forward (LIBOR) rates and the corresponding zero-coupon bond prices is:

$$F_i(t) = \frac{1}{\delta}\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right),$$

where $P(t, T)$ denotes zero-coupon bond price at time t with maturity T (ZCB pays 1 unit of currency at time T)

In general LMM model:

- Each forward F_i is modelled as process $F_i(t)$.
- Dynamics of the forward process driven by an N-dimensional correlated Wiener process $W_1(t), \dots, W_N(t)$
- Let ρ_{ij} be correlation coefficient between $W_i(t)$ and $W_j(t)$:

$$dW_i(t)dW_j(t) = \rho_{ij}(t)dt$$

LIBOR MARKET MODEL (LMM) FOR YIELD CURVES

Lognormal LIBOR Market model:

$$dF_i(t) = \mu_i(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i(t) \quad i = 1, \dots, N$$

That is, we model the set of equations:

$$\begin{aligned}
dF_1(t) &= \mu_1(t)F_1(t)dt + \sigma_1(t)F_1(t)dW_1(t) \\
dF_2(t) &= \mu_2(t)F_2(t)dt + \sigma_2(t)F_2(t)dW_2(t) \\
&\vdots \\
dF_N(t) &= \mu_N(t)F_N(t)dt + \sigma_N(t)F_N(t)dW_N(t)
\end{aligned}$$

Process $F_i(t)$ gets killed (expires) at $t = T_i$ as the forward rate fixes.

KB Pojišťovna

LIBOR MARKET MODEL (LMM) FOR YIELD CURVES

It can be shown that the drift of risk-neutral process can be written as:

$$dF_i(t) = F_i(t)\left(\sum_{j=m(t)}^{i} \frac{\delta F_j(t)\sigma_i(t)\sigma_j(t)\rho_{ij}(t)}{1+\delta F_j(t)}\right)dt + F_i(t)\sigma_i(t)dW_i(t) \quad i = 1, \dots, N,$$

where $m(t)$ is the index of the first forward rate that has not expired after t.

=> whole model completely defined if we know the volatility functions σ_i and the correlation matrix ρ_{ij}.

Modelling of N Brownian motions practically not feasible, instead process driven by small number of independent Brownian motions:

$$\frac{dF_i(t)}{F_i(t)} = drift + \sum_{q=1}^{D} \sigma_i^q(t)dZ^q(t) \quad i = 1, \dots, N,$$

where $D \in \{1,2,3,4\}$ and Z^q are independent Brownian motions.

...factors' reduction

LIBOR MARKET MODEL (LMM) FOR YIELD CURVES

By setting $D = 2$ and assuming volatility components in the form of $\sigma_i^q(t) \rightarrow v(t)\sigma_i^q$, we can write:

$$\frac{dF_i(t)}{F_i(t)} = drift \ + v(t) \sum_{q=1}^{2} \sigma_i^q dZ^q(t) \quad i = 1, \dots, N.$$

Model implemented by applying Itô's lemma on $\log(F_i(T_{k+1}))$ - model time step equal to δ:

...discretization

$$F_i(T_{k+1})$$
$$= F_i(T_k) exp\left[\sum_{j=k+1}^{i} \frac{\delta F_j(T_k) v^2(T_k) \sum_{q=1}^{2} \sigma_{j-k}^q \sigma_{i-k}^q}{1+\delta F_j(T_k)} \delta - \frac{1}{2} v^2(T_k) \sum_{q=1}^{2} {\sigma_{i-k}^q}^2 \delta + v(T_k)\left(\sigma_{i-k}^1 \sqrt{\delta} \varepsilon_1 + \sigma_{i-k}^2 \sqrt{\delta} \varepsilon_2\right)\right],$$

$\varepsilon_1, \varepsilon_2$ are independent normal variables

Note: The process for $F_i(t)$ cancels at $t = T_i$ since the i-th forward settles at T_i. Duration of the modelled curve decreases by δ over each model time-step.

KB
Pojišťovna

LIBOR MARKET MODEL (LMM) FOR YIELD CURVES

What's important: Volatility and correlation structure of the model is characterized by volatility factors σ_i^q and volatility scaling factor $v(t)$.

Parameters of the model to be calibrated:

$F_i(0)$ … initial forward curve

σ_i^1, σ_i^2 … volatility factors for each time interval, $i = 1, \dots, N$

$v(t)$ … volatility scaling factor (deterministic function)

FROM LMM TO LMM+

Moving to even more complex model:

…by introducing a displacement parameter θ and a stochastic variance $v(t)$

Displacement:

- Allows negative rates
- Reduces occurrence of exploding rates
- Introduces more volatility when rates are low

Stochastic volatility:

- More realistic description of interest rate volatility

FROM LMM TO LMM+

LMM+ risk neutral model:

$$\frac{dF_i(t)}{F_i(t)+\theta} = drift\ + \sqrt{v(t)} \sum_{q=1}^{2} \sigma_i^q Z^q(t) \quad i = 1, \ldots, N,$$

with the $drift$ in the form of

$$v(t)\left(\sum_{j=m(t)}^{i} \frac{\delta(F_k(t)+\theta)}{1+\delta F_k(t)} \sum_{q=1}^{2} \sigma_i^q \sigma_j^q\right) dt$$

$v(t)$ is an additional random process driven by Cox-Ingersoll-Ross (CIR) model:

$$dv(t) = a\big(b - v(t)\big)dt + \varepsilon\sqrt{v(t)}dW(t)$$

Correlation structure between process $v(t)$ and forward rate depends further on a correlation parameter ρ

LMM+

Parameters of the model to be calibrated:

$F_i(0)$ … initial forward curve

θ … displacement factor

σ_i^1, σ_i^2 … volatility factors for each time interval $[T_{i-1}, T_i], i = 1, \dots, N$

$v(0)$ … initial value of volatility scaling factor

ρ … correlation parameter

a, b, ε … factors of the CIR model

Calibration = Calculation of initial parameters of the model to be consistent with the current market situation (numerical methods)

= we want that calculated values of bonds or swaptions from the model agree on the current prices.

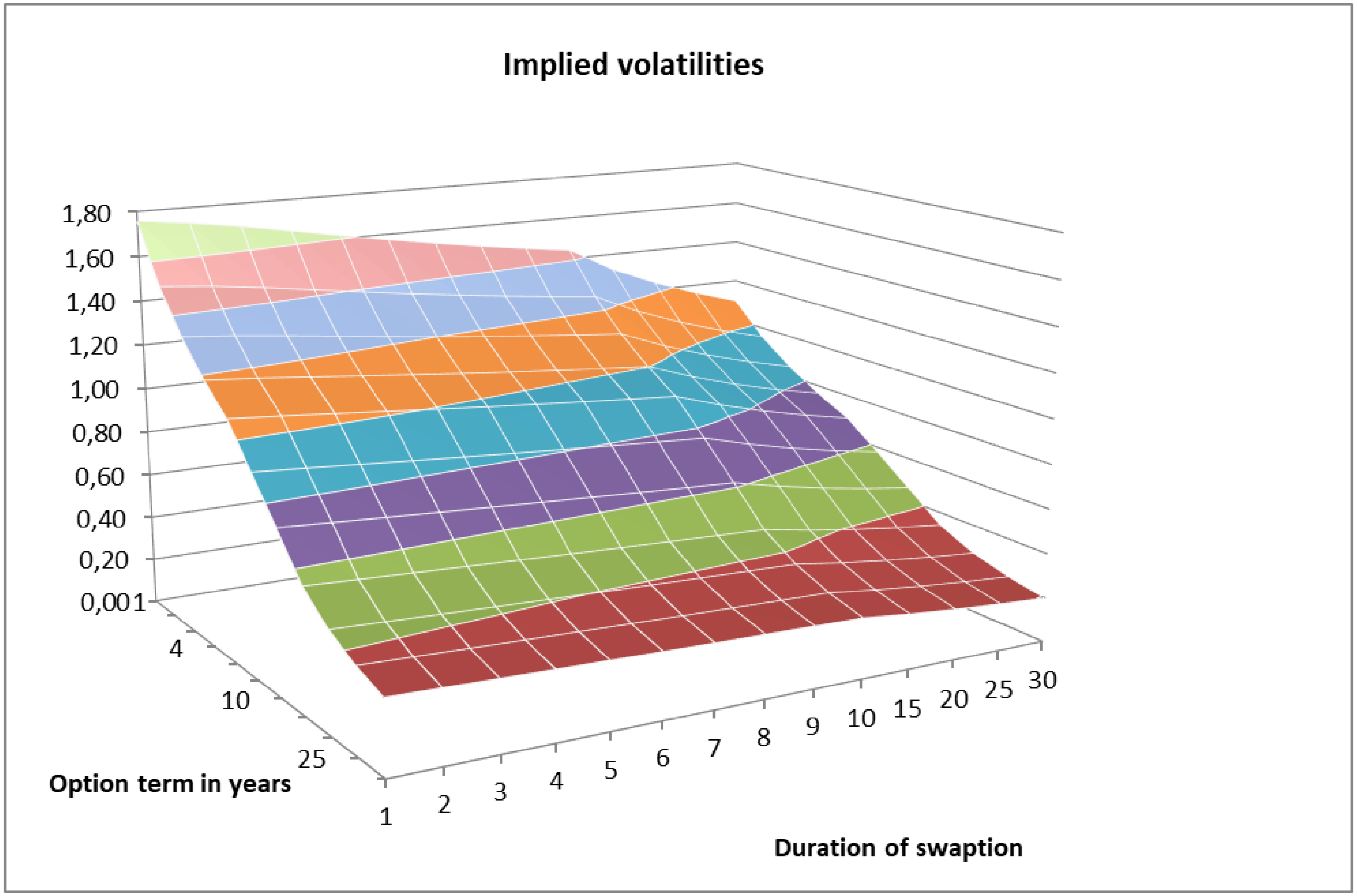
LMM+ MARKET CONSISTENT CALIBRATION

How calibration of LMM+ can look like :

1. Initial forward curve $F_i(0)$ - derived from government bond prices or swap rates (e.g., by Nelson-Siegel method)
2. Displacement θ – set to a reasonable value
3. Factors σ_i^1, σ_i^2- calibrated to empirical correlations estimated from historical forward rates
4. Market swaption prices and implied volatilities – estimated from market data
5. Initial value of volatility scaling factor $v(0)$ - reasonably set
6. Other parameters ρ, a, b, ε fitted numerically so that volatilities calculated from the model correspond to the market implied volatilities (optimization algorithms)

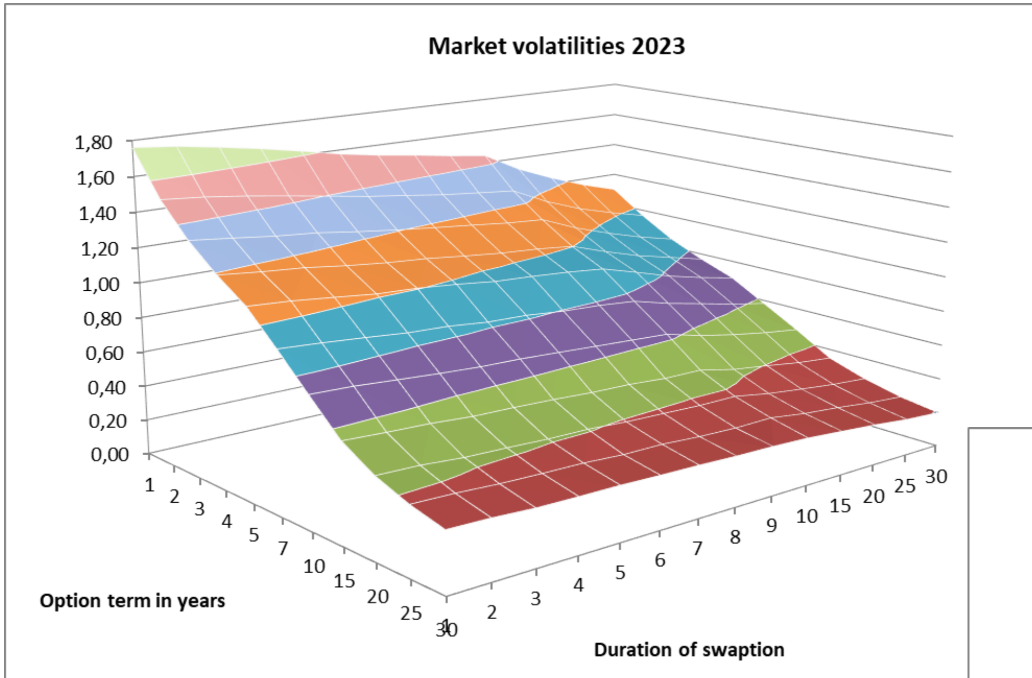
…fitting process

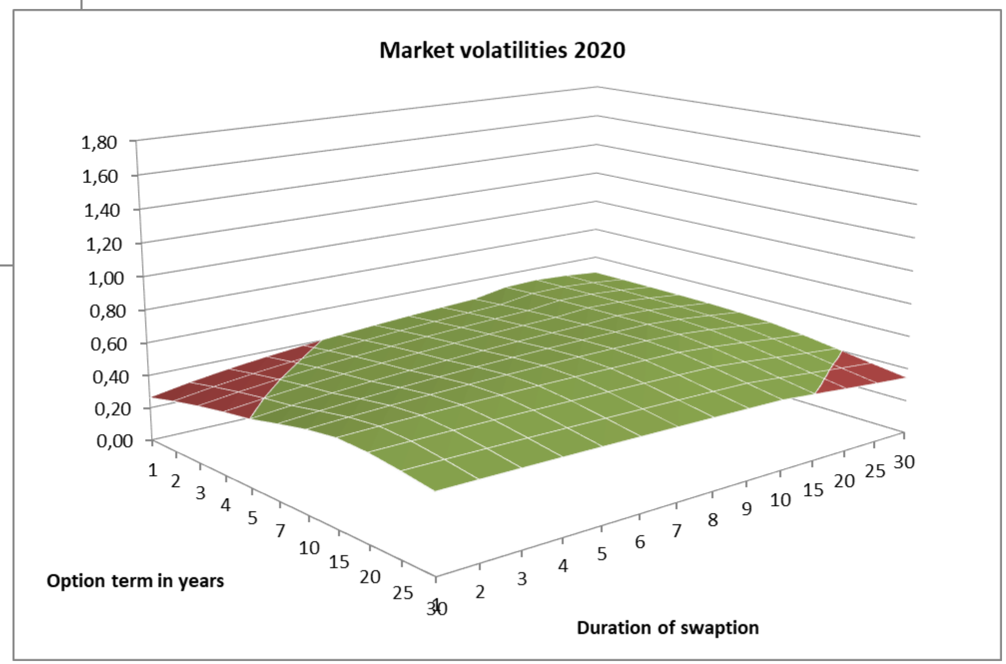
SWAPTION IMPLIED VOLATILITIES EXAMPLE



Interal data

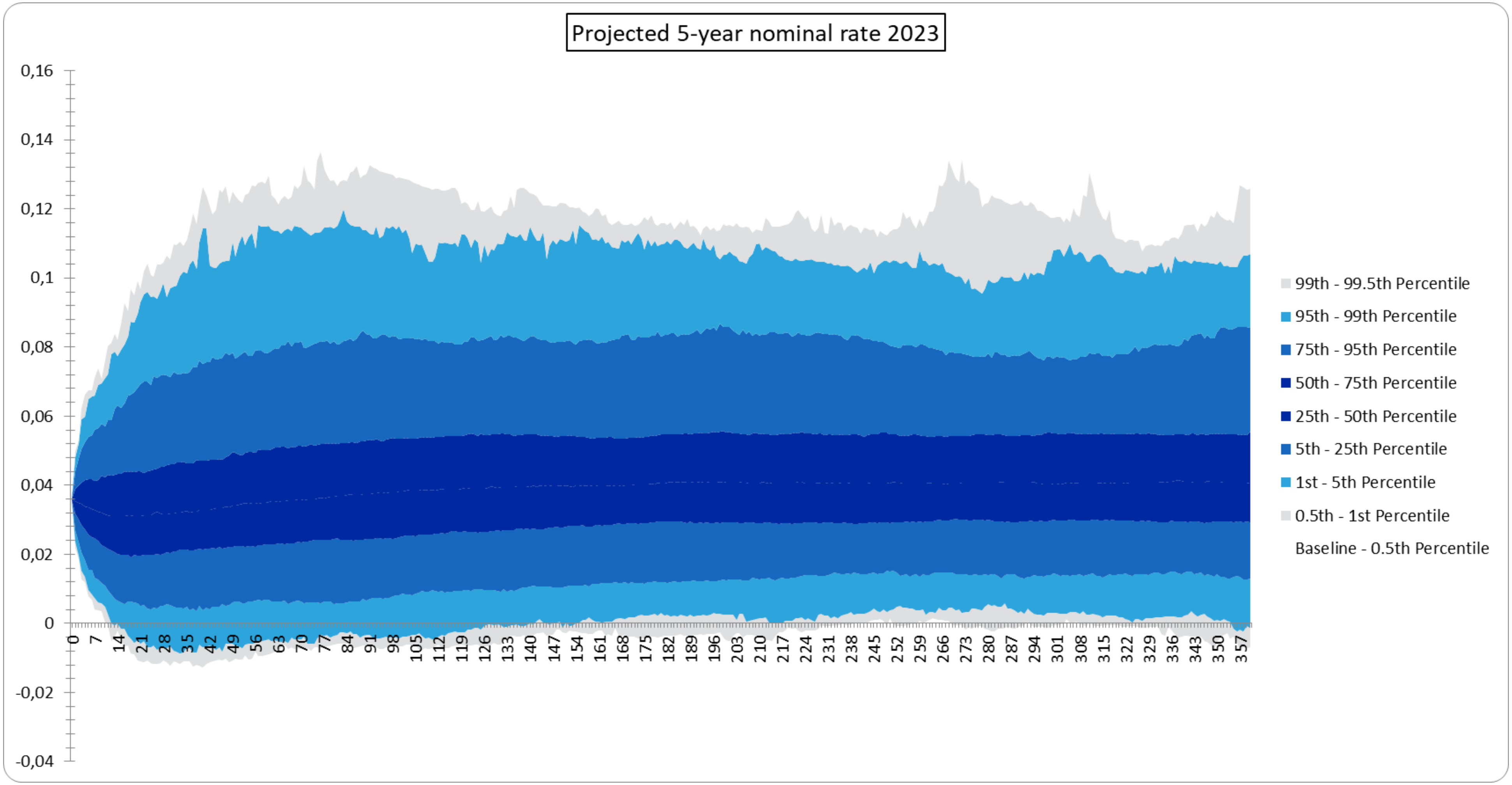
SWAPTION IMPLIED VOLATILITIES EXAMPLE





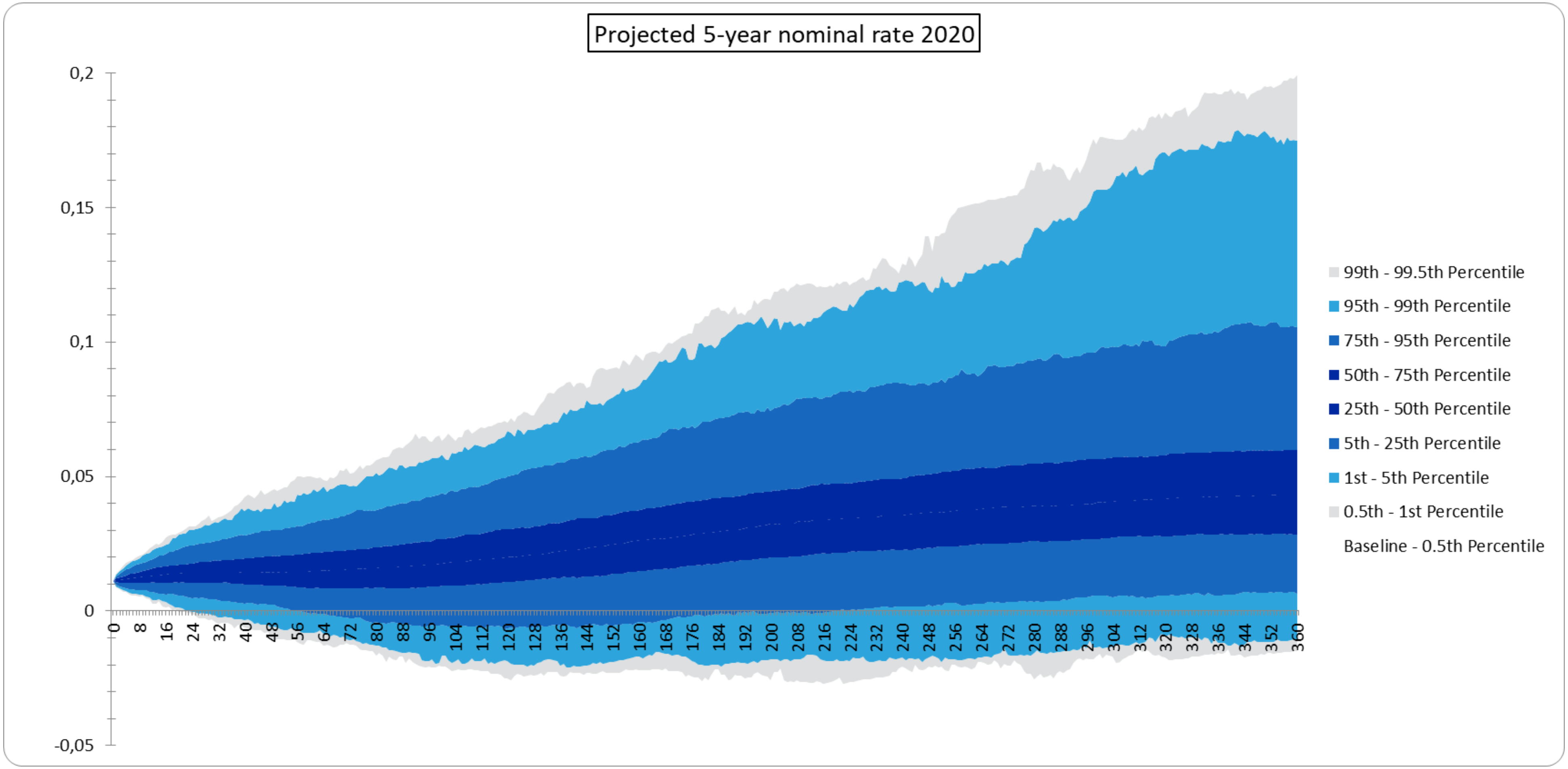
Interal data

SWAPTION IMPLIED VOLATILITIES EXAMPLE



Interal data

SWAPTION IMPLIED VOLATILITIES EXAMPLE



Interal data

REAL INTEREST RATE - TWO FACTOR VASICEK

Example of short-rate model – specified by stochastic behavior of the short-term interest rate (and other stochastic variable potentially)

$$dr(t) = a_r\big(m(t) - r(t)\big)dt + \sigma_r dW_r(t)$$

$$dm(t) = a_m\big(\mu - m(t)\big)dt + \sigma_m dW_m(t),$$

where:

$m(t)$ … process of mean-reversion level

$a_r, a_m, \sigma_r, \sigma_m, \mu$ are parameters of the model

W_r and W_m are Brownian motions

Under these assumptions, distribution of $r(t)$ is normal … allowing negative real interest rates.

*

REAL INTEREST RATE - TWO FACTOR VASICEK

Entire term structure can be calculated. From the risk-neutral price of an inflation-linked zero-coupon bond:

$$P(t,T) = E\left[exp\left(-\int_t^T r(t)dt\right)\right]$$

This expression can be calculated analytically.

Calibration of the model:

$r(0), m(0)$ … initial values, derived from current short and long term rates and inflation forecast

$a_r, a_m, \sigma_r, \sigma_m, \mu$ … numerical calculation so that modelled bond prices correspond to the market ones

INFLATION

Inflation derived from the nominal and real interest-rate curves.

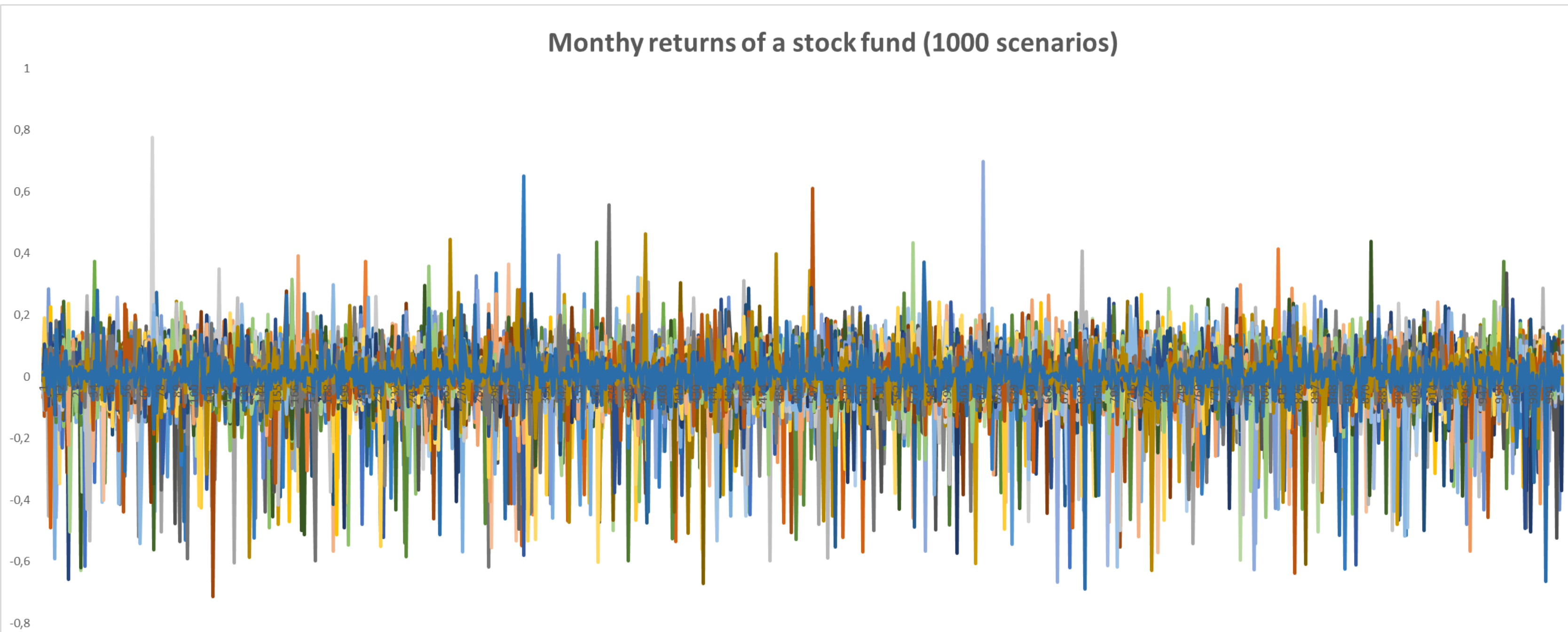
The inflation rate as the difference between the nominal short rate and the average real short rate:

$$Rate(t) = exp\left(Nominal.ShortRate(t - \Delta t) - \frac{1}{2}\left(Real.ShortRate(t - \Delta t) + Real.ShortRate(t)\right)\right) - 1$$

SVJD MODEL FOR EQUITIES

SVJD = Stochastic volatility jump diffusion model

- Combination of the stochastic volatility Heston model and the Merton jumps model
- Sudden large movements ("jumps") observed in practice
- Occurrence of jumps described by a Poisson random variable



SVJD MODEL FOR EQUITIES

Risk neutral model:

$$\frac{dS(t)}{S(t)} = (r(t) - \lambda\bar{\mu})dt + \sqrt{v(t)}dW_1(t) + (J-1)dN(t),$$

where variance $v(t)$ is a random process described by CIR model:

$$dv(t) = a\big(b - v(t)\big)dt + \varepsilon\sqrt{v(t)}dW_2(t)$$

W_1 and W_2 are correlated Wiener processes.

$N(t)$ … random number of jumps on the interval $[0, t]$, i.e. Poisson process: $N(t) \sim Poisson(\lambda t)$

$J \sim LogNormal(\mu_J, \sigma_J^2)$, i.e. $\ln(J) \sim N(\mu_J, \sigma_J^2)$ … jumps are log-normally distributed

$\bar{\mu} = \exp\left(\mu_J + \frac{\sigma_J^2}{2}\right) - 1 = E[J] - 1$

KB Pojišťovna

SVJD MODEL FOR EQUITIES

SVJD model

- Stochastic volatility
- Impact of price jumps via lognormal process + number of occurrences of jumps by a Poisson process

Calibration example

Calibration of the parameters of the SVJD model based on implied volatilities from options for the currency of the equity index.

… optimization problem to fit the model option prices to those observed on the market

CHARACTERISTICS ESG MUST DEAL WITH

Shock generation framework

$$RandomNumber_i \sim Uniform\,(0,1)$$

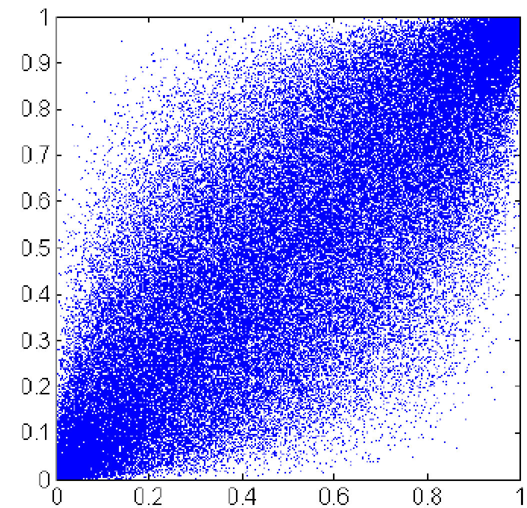
- For Monte Carlo method
- Pseudo-random number generation
- Specification of probability measure under which stochastic model dynamic is simulated (user can either choose risk-neutral or real-world)

Dependence structure of variables modelled

- Choice of a copula used for dependence structure – e.g. Gaussian copula or T-copula

Calibration of correlation (between asset classes)

- Historical correlations or Monte Carlo simulation



*

2

VALIDATION OF SCENARIOS

Examples

KB Pojišťovna

VALIDATIONS

As the number of simulation scenarios is finite (and numerical approximations used), it is important to test the desired property of the generated scenarios for both

- market-consistency, and
- risk-neutrality

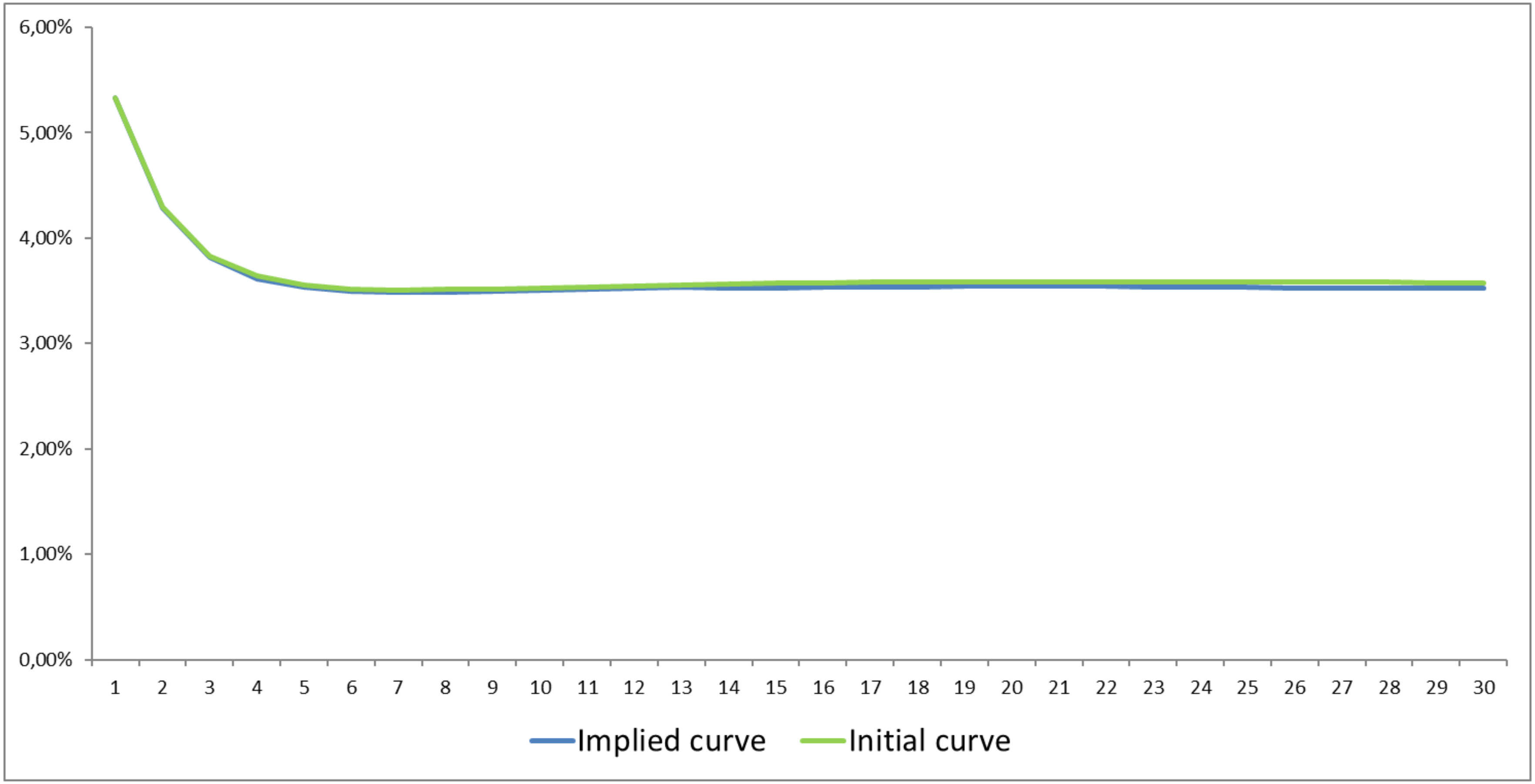
Inputs to validations: generated scenarios

Goal:

- To check that calculated prices and yields from generated scenarios agree (on average) to calibration targets
- To check that the present value of each asset class modelled equals the initial value (martingale property)

INITIAL YIELD CURVE

= test that model implied yield curve agrees on the initial yield curve

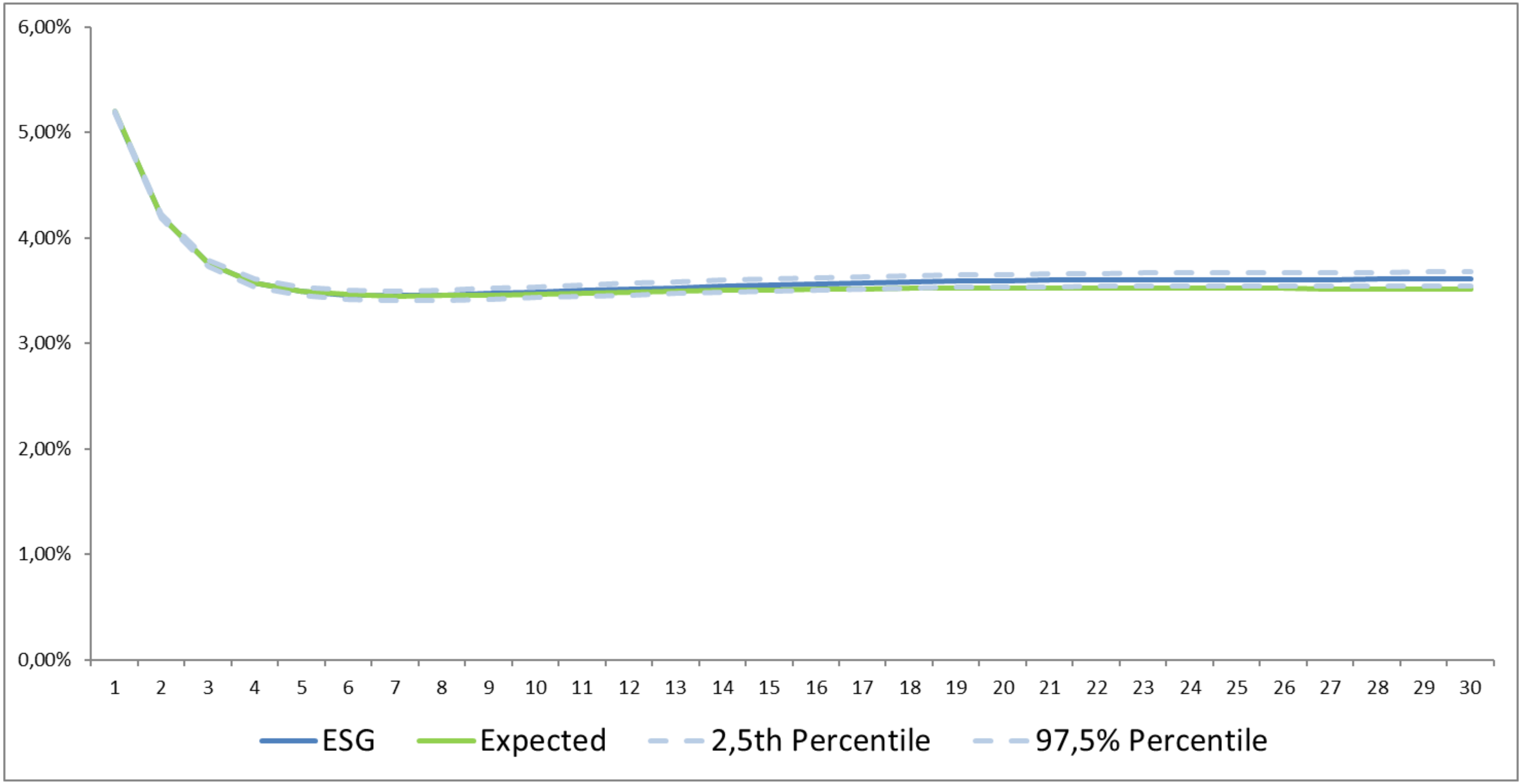


*

KB Pojišťovna

ZERO-COUPON BONDS YIELDS

= validates that the calculated zero-coupon yields corresponds to the zero-coupon yields from the initial curve



KB Pojišťovna

MARTINGALE TEST

- To check the present value of each asset class modelled equals the initial value (martingale property).
- To validate whether the number of scenarios is sufficient, i.e. the sampling error is sufficiently small, we can test that each asset earns the same as the cash
- Let S be total return index of an asset, C total cash return. It should hold for any projection time T:

$$E\left[\frac{1}{C(T)}S(T)\right] = 1$$

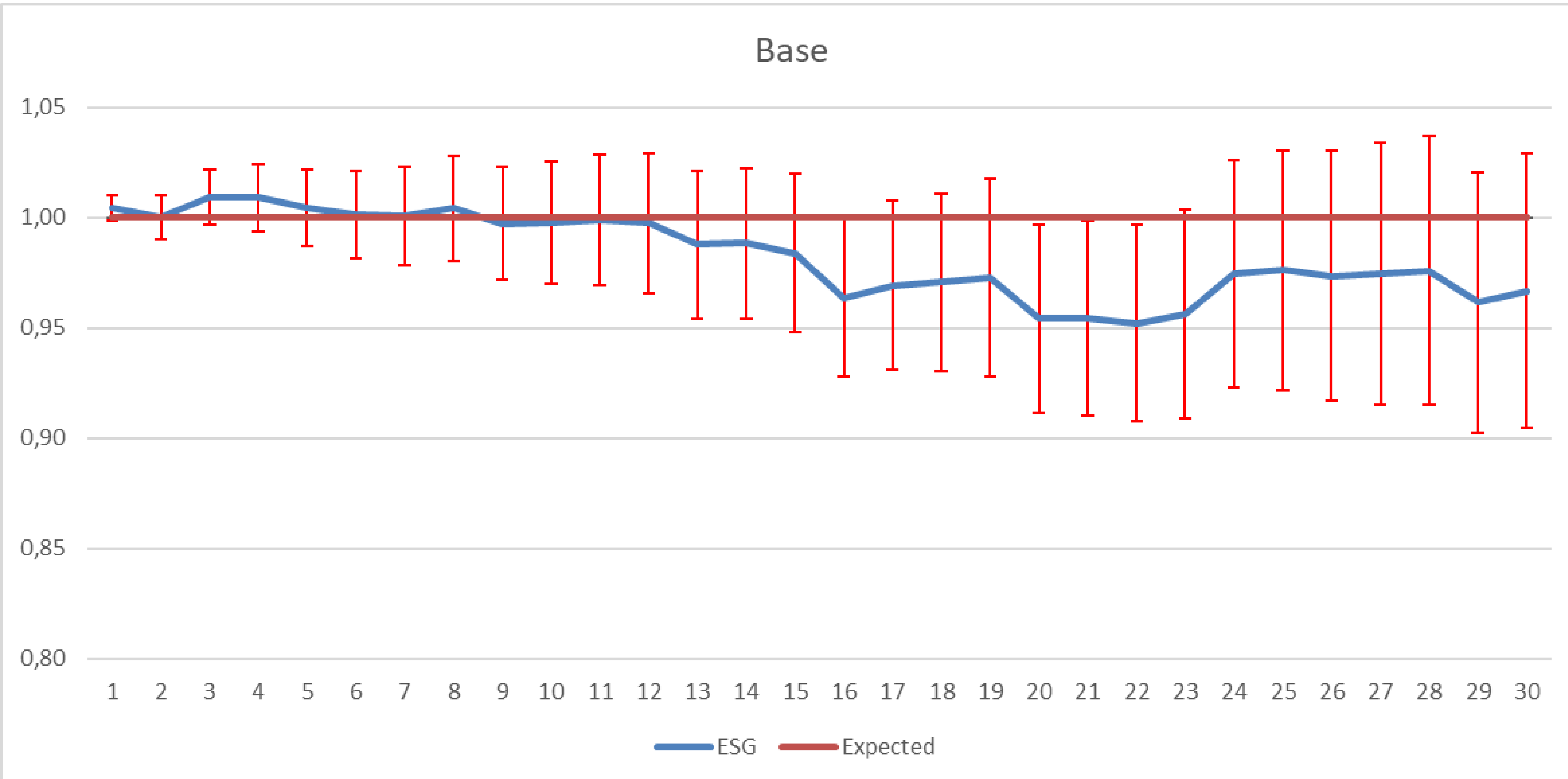
Calculation of mean (sample average) across all scenarios for each projection time-step

=> construction of 95% confidence intervals as:

$$(Mean \pm 1.96\ standard\ error)$$

EQUITY PRICES

To check that the present value of each asset class modelled equals the initial value (martingale property)



INTEREST RATE VOLATILITIES

To validate volatilities implied by the ESG output

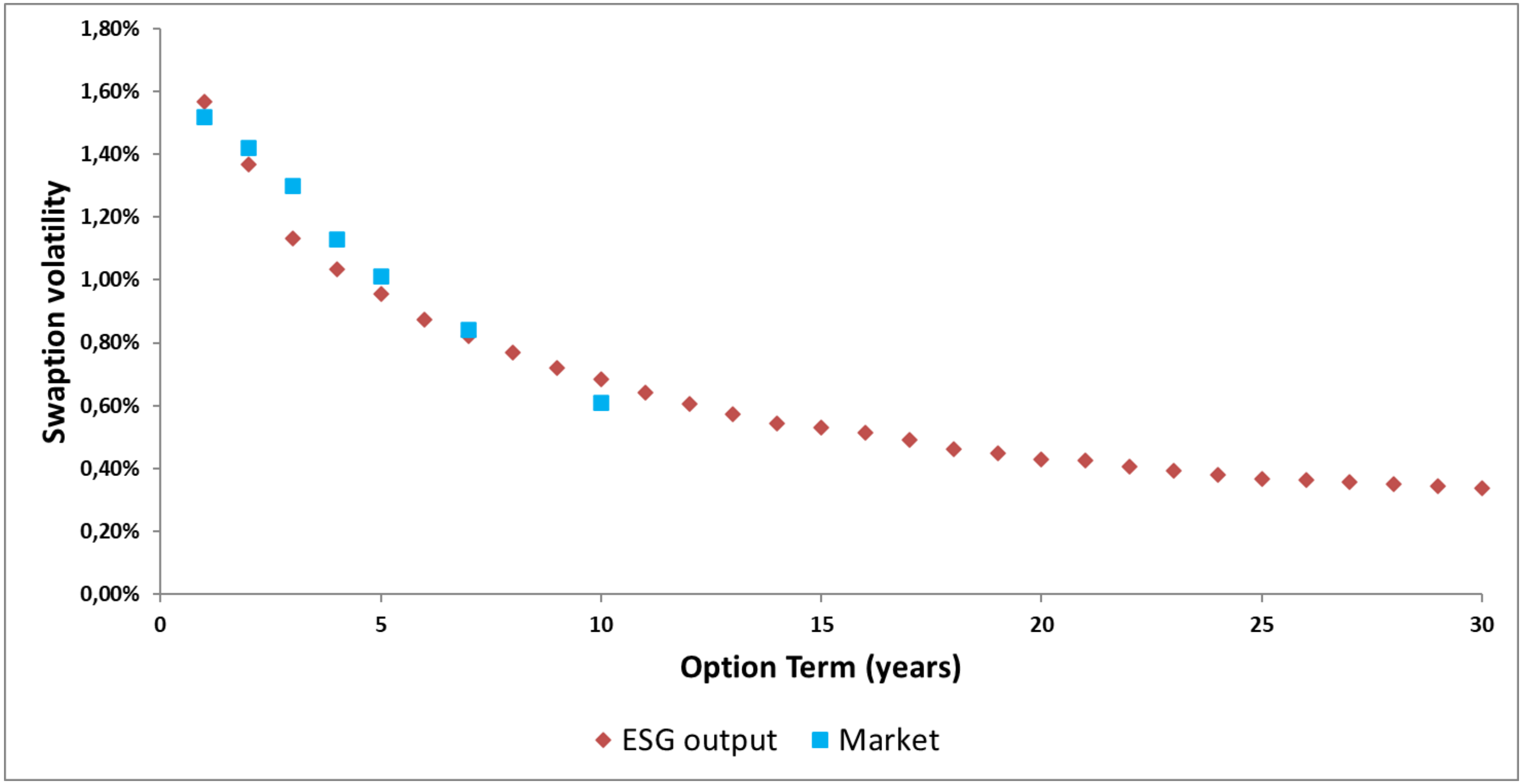
- Calculate swaption prices from generated scenarios – estimation based on the mean from the scenarios

=average discounted swaption payoff

- Calculate implied volatilities from the estimated swaption prices – e.g. using Black-Scholes or Bachelier formula for swaption
- Compare the volatilities implied from the scenarios to volatilities implied from the market prices of swaptions

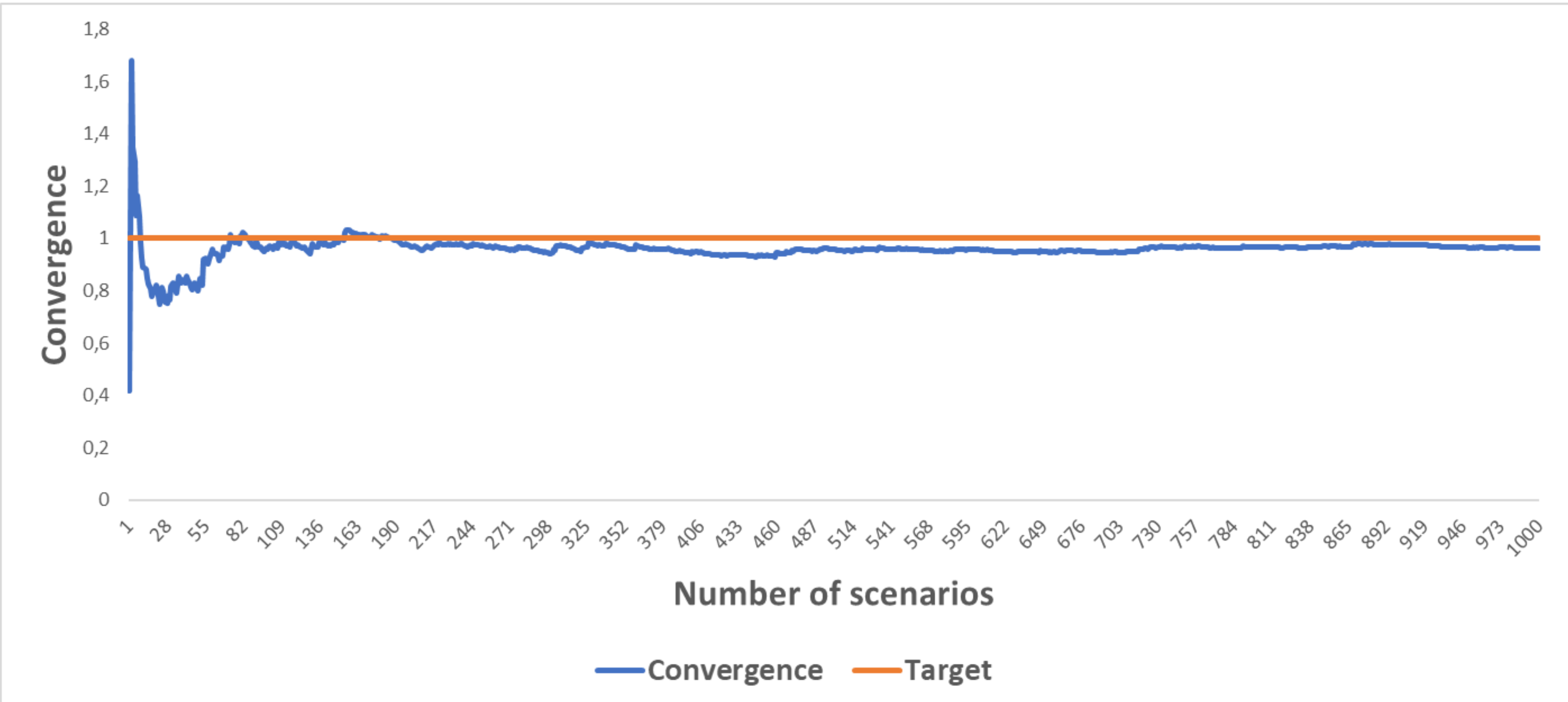
SWAPTION IMPLIED VOLATILITIES EXAMPLE

Example of volatilities estimated from model to market implied volatilities (5Y swaption):



CONVERGENCE TEST

- Alternative expression of martingale test
- Validates convergence of asset returns to the martingale property over the number of scenarios



3

CZECH MARKET

MARKET RESEARCH

- 8 biggest insurance companies that uses ESG on Czech market were surveyed
- However, only 4 responded => probably no generalization should be applied ☺

1. **Are ESG scenarios provided to you from the group, or do you use ESG internally?**
   - *1 internally, 3 have RN scenarios from the group (of which 1 generates RW scenarios internally)*
2. **How many scenarios do you use?**
   - *All 1000 scenarios for BEL*
3. **What model do you use for nominal rates simulation?**
   - *2 LMM+ model, 1 LMM model, 1 Hull-White model + Gaussian two factor model*
4. **Do you use real-world scenarios?**
   - *2 don’t*
   - *1 for ALM*
   - *1 only as single scenarios (e.g. for VNB, planning, LAT)*

LITERATURE

Bates, D., Jumps and Stochastic Volatility (1996) : Exchange Rate Processes Implicit in Deutsche Mark Options, The Review of Financial Studies, Vol. 9, No. 1, pp. 69-107

Brigo, D., and Mercurio, F. (2006) : Interest Rate Models - Theory and Practice, Springer Verlag

Calabrese, G. (2011): LMMPlus - Model Definition. Barrie&Hibbert

Hammond, G. (2014): LMM Plus Nominal interest rate modelling: MC Technical Summary. Moody's Analytics

Hull, J., White, A. (1999): Forward rate volatilities, swap rate volatilities and the implementation of the Libor Market Model

Rebonato, R. (2002): Modern pricing of interest-rate derivatives: the LIBOR market model and beyond. Princeton, NJ [u.a.]: Princeton Univ. Press

Thank you!

bour@komercpoj.cz